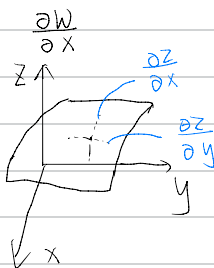




猴难思考

多元函数

偏导数



$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{z(x+\Delta x, y) - z(x, y)}{\Delta x}$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{z(x, y+\Delta y) - z(x, y)}{\Delta y}$$

$$\textcircled{2} \frac{\partial u}{\partial x} = \frac{u}{x} x^{\frac{u}{x}-1}$$

$$\frac{\partial u}{\partial y} = x^{\frac{u}{x}} \ln x \cdot \frac{1}{x} \quad (z^x)' = z^x \ln x$$

$$\frac{\partial u}{\partial z} = x^{\frac{u}{x}} \ln x \cdot (-\frac{u}{z}) \quad (z^x)' = z^x \ln x \cdot \frac{1}{z}$$

④ 先代入 $y=1 \therefore y'=0$

$z=x$ (不能代 x 因为 x 为 x 偏导数)

$$\frac{\partial z}{\partial x} = 1$$

○ 高阶偏导数(混合偏导): 例:

$$z = x^4 + y^4 - 4x^2y^2$$

$$\text{先对 } x \text{ 求导 } \frac{\partial z}{\partial x} = 4x^3 - 8xy^2$$

$$\text{再对 } y \text{ 求导 } \frac{\partial^2 z}{\partial x \partial y} = -16xy \quad (\text{先 } x \text{ 后 } y)$$

$$\text{若先对 } y \text{ 求导 } \frac{\partial z}{\partial y} = 4y^3 - 8x^2y$$

$$\text{再对 } x \text{ 求导 } \frac{\partial^2 z}{\partial y \partial x} = -16xy \quad (\text{先 } y \text{ 后 } x)$$

两者结果相同, 不论先对谁求导结果都相同

○ 一元函数: $y=f(x)$ 可导 $\Rightarrow$ 连续

多元函数: $z=f(x,y)$ 连续 $\nRightarrow$ 有偏导

$$\text{不连续, 但有偏导: } f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

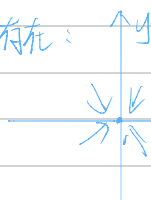
$$\text{不连续: } \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2} \xrightarrow{y=bx} \frac{b}{1+b^2}$$

$$\text{偏导存在: } \frac{\partial z}{\partial x}(0,0) = 0$$

$$(\because y=0 \therefore =0)$$

$$\frac{\partial z}{\partial y}(0,0) = 0$$

$$(\because x=0 \therefore =0)$$



则 y 轴

$$z = f(x,y) = x^2 + 3xy + y^2 \text{ 在 } (1,2) \text{ 处的偏导数}$$

$$\frac{\partial z}{\partial x} = 2x + 3y \quad \therefore \frac{\partial z}{\partial x}(1,2) = 8$$

$$\frac{\partial z}{\partial y} = 3x + 2y \quad \frac{\partial z}{\partial y}(1,2) = 7$$

$$\textcircled{1} S = \frac{u^2 + v^2}{uv}$$

$$\textcircled{2} z = (1+xy)^y$$

$$\textcircled{3} u = x^{\frac{y}{x}}$$

$$\textcircled{4} z = x + (y-1) \arcsin \sqrt{y}, \text{ 求 } \frac{\partial z}{\partial x}(0,1)$$

$$\textcircled{1} S = \frac{u^2 + v^2}{uv} = \frac{u}{v} + \frac{v}{u}$$

$$\frac{\partial S}{\partial u} = \frac{1}{v} - \frac{v}{u^2}$$

$$\frac{\partial S}{\partial v} = -\frac{u}{v^2} + \frac{1}{u}$$

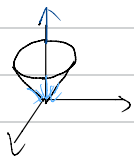
$$\textcircled{2} \frac{\partial z}{\partial x} = y(1+xy)^{y-1} \cdot y = y^2(1+xy)^{y-1}$$

当对 y 求导时, 为幂指函数, 须改取 $e^{y \ln(1+xy)}$

$$\therefore \frac{\partial z}{\partial y} = e^{y \ln(1+xy)} \cdot [\ln(1+xy) + \frac{xy}{1+xy}]$$

$$(a^b = e^{b \ln a})$$

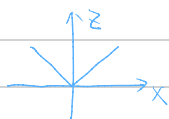
连续但不偏导数: $z = \sqrt{x+y^2}$



(0,0) 连续

$$\lim_{(x,y) \rightarrow (0,0)} \sqrt{x+y^2} = f(0,0)$$

但 $\frac{\partial z}{\partial x}(y=0) z = \sqrt{x} = |x|$



左右导数不一致
 $\therefore \frac{\partial z}{\partial x}$ 不存在
 同理 $\frac{\partial z}{\partial y}$ 不存在

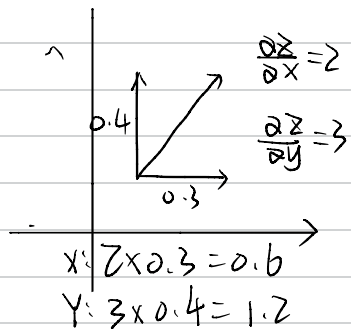
$$z = \sqrt{1+t^2} x$$

$$\frac{f(0+\Delta x, 0) - f(0, 0)}{\Delta x}$$

$$\frac{\sqrt{x^2}}{\Delta x} = \frac{\Delta x}{\Delta x} = 1$$

○

$$z = F(x, y) \begin{cases} \text{全微分: } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ \text{方向导数: } \frac{\partial z}{\partial l} = \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \cos \beta \\ \text{梯度: (向量): } \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right) \end{cases}$$



$$\frac{\partial z}{\partial x} = 2$$

$$\frac{\partial z}{\partial y} = 3$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$\downarrow$ $\downarrow$ $\downarrow$
 z 的变化量 x 变 y 变

全微分的判定公式:

$$\frac{\text{实际商数差}}{\sqrt{(dx)^2 + (dy)^2}} = 0$$

$\frac{z(x+dx, y+dy) - z(x, y) - \left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \right)}{\sqrt{(dx)^2 + (dy)^2}} = 0$

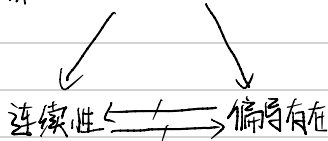
$z = e^{xy}$ 的全微分

$$\frac{\partial z}{\partial x} = y \cdot e^{xy} \quad \frac{\partial z}{\partial y} = x e^{xy}$$

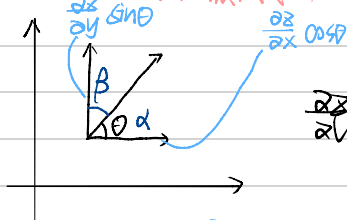
$$\Rightarrow dz = y e^{xy} dx + x e^{xy} dy$$

多元函数

偏导连续 $\rightarrow$ 可全微分



方向导数 (上坡下坡问题)



$$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \sin \alpha$$

$$dz = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$$

$$\frac{\partial z}{\partial l} = \frac{\frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$\begin{aligned} \text{(函数变化) / 步长} &= \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \cos \beta \\ &= \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta \end{aligned}$$

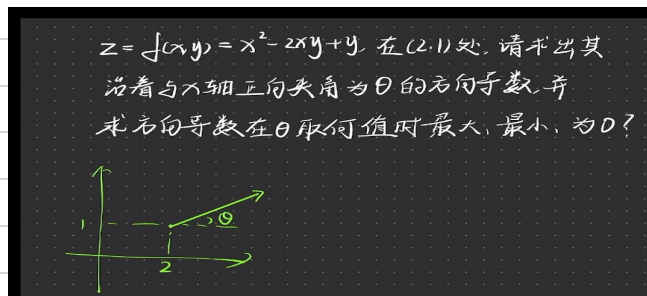
指向

$$\text{总结: } \frac{\partial z}{\partial l} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$$

$\downarrow$
 θ 与 $(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$ 同向, $\frac{\partial z}{\partial l}$ 最大 (上山最快)

θ 与 $(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$ 反向, $\frac{\partial z}{\partial l}$ 最小 (下坡最快)

θ 与 $(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$ 垂直, $\frac{\partial z}{\partial l}$ 为 0 (等高线)

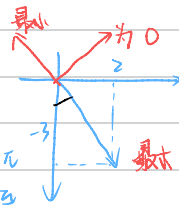


$$\frac{\partial z}{\partial l} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta$$

$$\frac{\partial z}{\partial x} = 2x - 2y, \quad \frac{\partial z}{\partial y} = -2x + 1$$

$$\frac{\partial z}{\partial x} (2,1) = 2, \quad \frac{\partial z}{\partial y} (2,1) = -3$$

$$\begin{aligned} \frac{\partial z}{\partial l} &= \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta \\ &= 2 \cos \theta - 3 \sin \theta \quad (\text{求其最大值}) \\ &= (2, -3) (\cos \theta, \sin \theta) \end{aligned}$$



$$\begin{cases} \theta \text{ 与 } (2, -3) \text{ 同向, 最大 } \theta = \arccos \frac{2}{3.5} \\ \theta \text{ 与 } (2, -3) \text{ 反向, 最小 } \theta = \arccos \frac{2}{3.5} + \pi \\ \theta \text{ 与 } (2, -3) \text{ 垂直, 为 0 } \theta = \arccos \frac{2}{3.5} + \frac{\pi}{2} \end{cases}$$

$z = x e^{2y}$ 在点 $(1,0)$ 处沿 $P(1,0)$ 到 $Q(2, \frac{1}{2})$ 的方向的方向导数

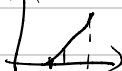
$$\text{总结: } \frac{\partial z}{\partial l} = \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \cos \beta$$



$$e^{2y}|_{(1,0)} = 1, \quad 2x e^{2y}|_{(1,0)} = 2$$

$$\cos \alpha = \frac{\sqrt{2}}{2}, \quad \cos \beta = -\frac{\sqrt{2}}{2}$$

$$\therefore \frac{\partial z}{\partial l} = -\frac{\sqrt{2}}{2}$$



$$\begin{aligned} \frac{\partial z}{\partial x} &= e^{2y}|_{(1,0)} \\ &= 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + \frac{1}{2} = \frac{3\sqrt{2}}{2} \end{aligned}$$

$$\frac{\partial z}{\partial y} = 2x e^{2y}|_{(1,0)} = 2 \cdot \frac{\sqrt{2}}{2}$$

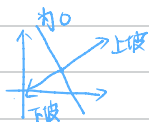
(与 x 轴正向夹角为 α)
 (与 y 轴正向夹角为 β)

$z = f(x, y) = \frac{1}{2}(x^2 + y^2)$, $P_0(2, 1)$ 处 (方向导数)

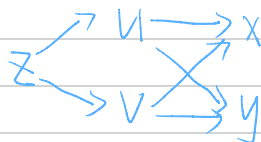
- ① 求 $f(x, y)$ 增大最快的方向 $(2, 1)$ 向量方向
- ② 减少最快的方向 $(-1, -2)$ 反向量方向
- ③ 方向导数为 0 的方向 $(-1, 2)$ 或 $(1, -2)$

$$\frac{\partial z}{\partial x}(2, 1) = 2 \quad \frac{\partial z}{\partial y}(2, 1) = 1$$

$\therefore$ 同成 $(2, 1)$



☆③ 画关系图



($\frac{\partial z}{\partial u}$ 代表 z 到 x 的偏导)

同理 $\frac{\partial z}{\partial v}$

同一链路的相乘

不同链路的相加

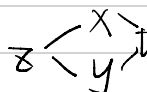
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= 2u \cdot 1 + 2v \cdot 1 = 2u + 2v = 4x$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= 2u \cdot 1 + 2v \cdot (-1)$$

$$= 2u - 2v = 4y$$



● 多元函数求导

例: $z = u^2 + v^2$, $u = x + y$, $v = x - y$

求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

① 代入: $z = (x + y)^2 + (x - y)^2$

$$= 2x^2 + 2y^2$$

$$\frac{\partial z}{\partial x} = 4x, \quad \frac{\partial z}{\partial y} = 4y$$

例: $z = e^{x-2y}$, $x = \sin t$, $y = t^3$ 求 $\frac{\partial z}{\partial t}$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= e^{x-2y} \cos t + e^{x-2y} \cdot (-2) \cdot 3t^2$$

(多个自变量写为 $\frac{\partial}{\partial x}$, 单个自变量 $\frac{d}{dt}$)

② 全微分形式不变性

$$z \begin{cases} (u, v) \\ (x, y) \end{cases} \quad dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$z du = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= dx + dy$$

同理可得 $dv = dx - dy$

$$\therefore dz = 2u(dx + dy) + 2v(dx - dy)$$

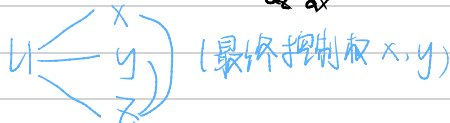
$$\text{变形} \Rightarrow (2u + 2v)dx + (2u - 2v)dy$$

$$\therefore \frac{\partial z}{\partial x} = 2u + 2v = 4x \quad \frac{\partial z}{\partial y} = 2u - 2v = 4y$$

例: $u = f(x, y, z) = e^{x^2 + y^2 + z^2}$

$$z = x^2 \sin y$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$



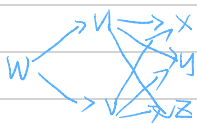
(最终控制 x, y)

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 2xe^{x^2 + y^2 + z^2} + 2ze^{x^2 + y^2 + z^2} \cdot 2x \sin y$$

全影响 直接影响

ps: u - 变量
 f - 表达式

例: $w = f(\underbrace{x+y+z}_u, \underbrace{xyz}_v)$, 求 $\frac{\partial w}{\partial x}$, $\frac{\partial^2 w}{\partial x \partial z}$



$$\frac{\partial w}{\partial x} = \underbrace{\frac{\partial f}{\partial u}}_{f'_1} \cdot \underbrace{\frac{\partial u}{\partial x}}_1 + \underbrace{\frac{\partial f}{\partial v}}_{f'_2} \cdot \underbrace{\frac{\partial v}{\partial x}}_{yz}$$

$$= f'_1 + f'_2 yz$$

$$\frac{\partial^2 w}{\partial x \partial z}$$

$$= f''_{11} + f''_{12} xy + f''_{22} y$$

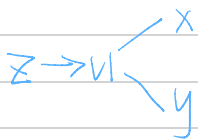
(f''_{11} = 对 u 求两次导, f''_{12} 先对 u 求一次 再对 z 求一次)

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial (f'_1)}{\partial z} + \frac{\partial (f'_2)}{\partial z} yz + f'_2 y$$

$$\begin{aligned} f'_1 &\rightarrow u \begin{cases} x \\ y \\ z \end{cases} = \frac{\partial f'_1}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f'_1}{\partial v} \cdot \frac{\partial v}{\partial z} + \left(\frac{\partial f'_2}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f'_2}{\partial v} \cdot \frac{\partial v}{\partial z} \right) yz + f'_2 y \\ &= f''_{11} \cdot 1 + f''_{12} xy + (f''_{21} \cdot 1 + f''_{22} \cdot xy) yz + f'_2 y \end{aligned}$$

(f'_2)

例: $z = f(\underbrace{x^2+y^2}_u)$ 求 $\frac{\partial z}{\partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$, $\frac{\partial^2 z}{\partial y^2}$



$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} \\ &= f'_u \cdot 2x \\ &= f'_u (x^2+y^2) 2x \end{aligned}$$

$$\begin{aligned} f'_u &\rightarrow u \begin{cases} x \\ y \end{cases} \frac{\partial z}{\partial x \partial y} = \frac{\partial f'_u}{\partial u} \cdot \frac{\partial u}{\partial y} 2x \\ &= f''_{uu} \cdot 2y 2x \\ &= 4xy f''_{uu} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} \\ &= f'_u \cdot 2y \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= \frac{\partial f'_u}{\partial u} \cdot \frac{\partial u}{\partial y} 2y + 2f'_u \\ &= f''_{uu} \cdot 2y \cdot 2y + 2f'_u \\ &= 4y^2 f''_{uu} + 2f'_u \end{aligned}$$

$$z \rightarrow u \begin{cases} x \\ y \end{cases}$$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} \\ &= f'_u 2x \end{aligned}$$

$$f'_u \rightarrow u \begin{cases} x \\ y \end{cases}$$

$$\begin{aligned} \frac{\partial z}{\partial y \partial x} &= \frac{\partial f'_u}{\partial u} \cdot \frac{\partial u}{\partial y} 2x \\ &= f''_{uu} 2y \cdot 2x \\ &= 4xy f''_{uu} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} \\ &= f'_u \cdot 2y \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= \frac{\partial f'_u}{\partial u} \cdot \frac{\partial u}{\partial y} 2y + 2f'_u \\ &= f''_{uu} \cdot 2y^2 + 2f'_u \end{aligned}$$

$z = f(x, y)$, 具有偏导数 $f'_x = 0$ $f'_y = 0$

$f''_{xx} = A$ $f''_{xy} = B$ $f''_{yy} = C$

① $AC - B^2 > 0$, 是极值点 $\begin{cases} A > 0 \text{ 极小} \\ A < 0 \text{ 极大} \end{cases}$

② $AC - B^2 < 0$ 不是极值

③ $AC - B^2 = 0$

例: $f(x, y) = x^3 - y^3 + 3x^2 + 3y^2 - 9x$

① 找 $f'_x = f'_y = 0$

$\begin{cases} 3x^2 + 6x - 9 = 0 \\ -3y^2 + 6y = 0 \end{cases} \Rightarrow \begin{cases} x = 1, -3 \\ y = 0, 2 \end{cases}$

②

	$f''_{xx}(A)$	$f''_{xy}(B)$	$f''_{yy}(C)$	
$(1, 0)$	12	0	6	极小
$(-3, 0)$	-12	0	6	极大
$(1, 2)$	12	0	-6	
$(-3, 2)$	-12	0	-6	

$f''_{xx} = 6x + 6$
 $f''_{xy} = 0$
 $f''_{yy} = 6y + 6$

③

$f(1, 0) = 1 + 3 - 9 = -5$

$f(-3, 2) = -27 - 8 + 27 + 12 + 27 = 31$

$x = (\frac{1}{2})^0$
 $f'_x = 2x - 1$ $f'_y = 4y$ $f'_{xy} = 0$
 $f''_{xx} = 2$ $f''_{yy} = 4$
 $(\frac{1}{2}, 0)$

例: $T = x^2 + 2y^2 - x$ $\{(x, y) | x^2 + y^2 \leq 1\}$ 求最值

$\begin{cases} T'_x = 2x - 1 = 0 \\ T'_y = 4y = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = 0 \end{cases}$

$T''_{xx} = 2$ $T''_{yy} = 4$

$T''_{xy} = 0$

$AC - B^2 = 8 > 0$

又 $A > 0 \therefore (\frac{1}{2}, 0)$ 为极小值

$T(\frac{1}{2}, 0) = -\frac{1}{4}$

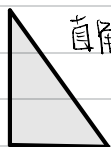
$\begin{cases} x^2 + y^2 = 1 \\ T = x^2 + 2y^2 - x \end{cases}$

$\Rightarrow T = -x^2 - x + 2(1 - x^2) = -3x^2 - x + 2$

$\therefore x$ 取 1 时为最大值

取 $-\frac{1}{2}$ 为最小值

$T(1, 0) = 1$ 极大
 $T(-\frac{1}{2}, 0) = \frac{5}{4}$ 极大
 $T(\frac{1}{2}, 0) = -\frac{1}{4}$ 极小



直角三角形, 斜边长为4, 问最大是?

$C = x + y + 4$ 最大值

$x^2 + y^2 = 16$ 条件

$$L(x, y) = x + y + 4 + \lambda(x^2 + y^2 - 16)$$

$$\begin{cases} L'_x = 0 \\ L'_y = 0 \\ L'_\lambda = 0 \end{cases} \Rightarrow \begin{cases} 1 + 2\lambda x = 0 \text{ ①} \\ 1 + 2\lambda y = 0 \text{ ②} \\ x^2 + y^2 = 16 \text{ ③} \end{cases}$$

$$\left(-\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 = 16 \quad (\text{将 } x, y \text{ 表示成 } \lambda \text{ 代入 ③})$$

$$\Rightarrow \lambda = \pm \frac{\sqrt{2}}{8}$$

$$x = -\frac{1}{2\lambda} = 2\sqrt{2} \quad (\lambda \text{ 不能取正值})$$

$$y = -\frac{1}{2\lambda} = 2\sqrt{2}$$

隐函数定理3:

$$\begin{cases} u + x + y = 3 \\ x + y + v = 4 \\ x - y - u = 5 \end{cases}$$

四个变量

$x \quad y \quad u \quad v$

但无论 u 取何值

即 x, y, v 的值都有控制

下来

$\therefore x, y, v$ 都受 u 的控制

$\therefore$ 只有 u 一个自变量而有三个函数

$$x = x(u) \quad y = y(u) \quad v = v(u)$$

隐函数定理

$$\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases} \Rightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

隐函数定理1. $F(x, y) = 0, F'_x, F'_y$

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y} (F'_y \neq 0)$$

例: $F(x, y) = x^2 + y^2 = 1$ 求 $\frac{dy}{dx}$

$$F'_x = 2x$$

$$F'_y = 2y$$

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y} = -\frac{x}{y}$$

求 $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$

$$\begin{cases} u + x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x} = 0 \\ y \frac{\partial u}{\partial x} + v + x \frac{\partial v}{\partial x} = 0 \end{cases}$$

(只有2个未知)

2个方程

补充克莱姆法则

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

$$\Rightarrow x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

隐函数定理2:

$$x + y + z - 2\sqrt{xy}z = 0$$

↓ 变化为看作 x, y 影响着 z 的函数

$$F(x, y, z) = x + y + z - 2\sqrt{xy}z$$

$$F'_x = 1 - \frac{yz}{\sqrt{xy}}$$

$$F'_y = 1 - \frac{xz}{\sqrt{xy}}$$

$$F'_z = 1 - \frac{x+y}{\sqrt{xy}}$$

$$\rightarrow \frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z}$$

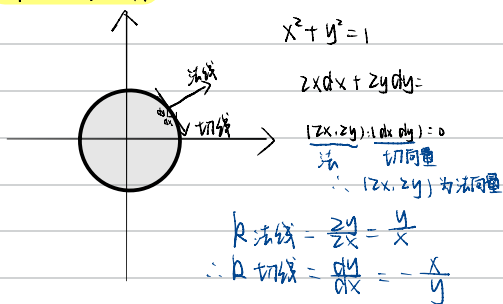
$$\frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -u-y \\ -v-x \end{vmatrix}}{\begin{vmatrix} x-y \\ y-x \end{vmatrix}} = \frac{-ux-yv}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} x-y \\ y-x \end{vmatrix}}{\begin{vmatrix} y-x \\ x-y \end{vmatrix}} = \frac{-xv+uy}{x^2+y^2}$$

同理可求 $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$

几何应用



空间曲线的切线, 法平面

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad t = t_0 \text{ 处坐标 } (x(t_0), y(t_0), z(t_0))$$

曲线上一分点

切向量 $(x'(t_0), y'(t_0), z'(t_0))$

$\therefore$ 切线: $\frac{x-x_{t_0}}{x'(t_0)} = \frac{y-y_{t_0}}{y'(t_0)} = \frac{z-z_{t_0}}{z'(t_0)}$

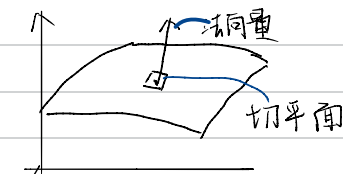
法平面: $x'(t_0)(x-x_{t_0}) + y'(t_0)(y-y_{t_0}) + z'(t_0)(z-z_{t_0}) = 0$

空间曲面

$$F(x, y, z) = 0$$

0 法向量 (F'_x, F'_y, F'_z)

切平面 $F'_x(x-x_0) + F'_y(y-y_0) + F'_z(z-z_0) = 0$



例: 求 $z = x^2 + y^2 - 1$ 在 $(2, 1, 4)$ 处切平面, 法线

$$F(x, y, z) = z - x^2 - y^2 + 1$$

$$F'_x = -2x \quad F'_y = -2y \quad F'_z = 1$$

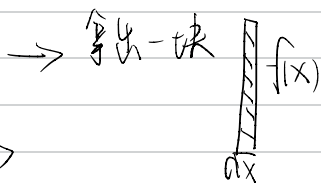
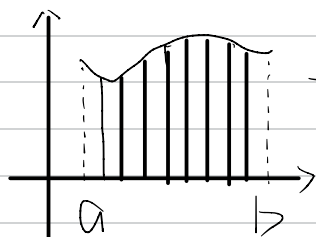
在 $(2, 1, 4)$ 处 $(F'_x, F'_y, F'_z) = (-4, -2, 1)$

切平面:

$$-4(x-2) - 2(y-1) + 1(z-4) = 0$$

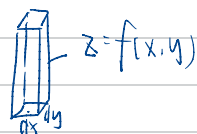
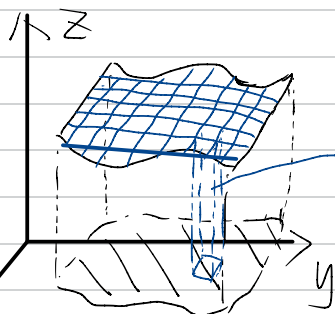
法线: $\frac{x-2}{-4} = \frac{y-1}{-2} = \frac{z-4}{1}$

重积分



$$\int_a^b f(x) dx = F(b) - F(a)$$

$$z = f(x, y)$$



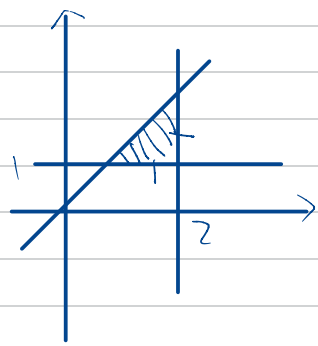
底面积: $dx dy$
高: $f(x, y)$

$$V = dx dy f(x, y)$$

$$\iint f(x, y) dx dy$$

两个积分符号

$\iint_D xy dx dy$, D 是由直线 $y=1, x=2, y=x$ 围成



解法1. $y=x$
 $y=1$

$$\int_1^2 \left[\int_1^x xy dy \right] dx$$

先坐标, $y_1 \rightarrow y_2$
y上积分
一个长条

积分函数 (题目已给)
积分 (对y)
x带

$$\frac{x}{2} y^2 \Big|_1^x = \frac{x^3}{2} - \frac{x}{2}$$

$$\int_1^2 \left(\frac{x^3}{2} - \frac{x}{2} \right) dx$$

$$\frac{x^4}{8} - \frac{x^2}{4} \Big|_1^2 = \frac{17}{8}$$

解法2

$$\int_{x_1=y}^{x_2=z} \int_y^z xy \, dx \, dy = \boxed{\int_1^2 dy \int_y^z xy \, dx} = \text{二次积分}$$

一个横条 $\square$

↓ 积分 (对 x , y 不变)

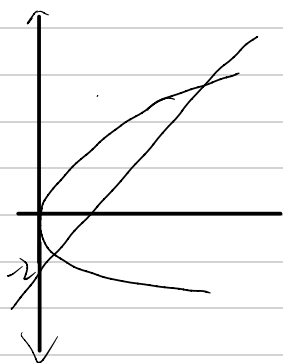
$$\frac{x^2 y}{2} \Big|_y^z = zy - \frac{y^2}{2}$$

$$= \int_1^2 (zy - \frac{y^2}{2}) \, dy$$

↓ 积分

$$y^2 - \frac{y^4}{8} \Big|_1^2 = \frac{9}{8}$$

$\iint_D xy \, d\sigma$, D 是 $y^2 = x$, $y = x - 2$ 围成的区域



- ① 画图
- ② 切割
- ③ = 二次积分
- ④ 计算

$y = x - 2 \quad z = 4$

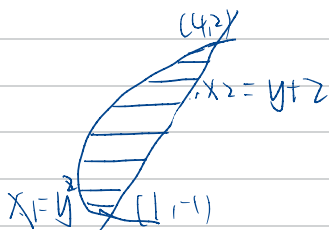
$y^2 = x$

左: $\int_1^4 \int_{x-2}^{\sqrt{x}} xy \, dy \, dx$ 右: $\int_1^4 \int_{x-2}^{\sqrt{x}} xy \, dy \, dx$

对于 y , (下端纵坐标) 积分

$x < 1 \quad y_1 = -\sqrt{x} \quad x \geq 1 \quad y_1 = x - 2$

左: $\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} xy \, dy \, dx$



$$\int_{-1}^2 \int_{y^2}^{y+2} xy \, dx \, dy$$

$$= \int_{-1}^2 \frac{y^2 x}{2} \Big|_{y^2}^{y+2} dy$$

$$= \int_{-1}^2 \left[\frac{y(y+2)^2}{2} - \frac{y \cdot y^4}{2} \right] dy$$

$$= \frac{1}{2} \left[\frac{y^4}{4} + \frac{4}{3} y^3 + 2y^2 - \frac{y^6}{6} \right] \Big|_{-1}^2$$

$$= \frac{45}{8}$$

$\iint_D e^{x^2} dx dy$, D 是 $y=x$, $y=0$, $x=1$ 围成区域



根: 若直接做 e^{x^2} 不会写原函数, 积不出来
但若竖切 就可以将 x 作为常数

$$\int_0^1 \int_0^x e^{x^2} dy dx$$

对 y 积分 不关 x 的事
∴ 很好解决难题

$$= \int_0^1 e^{x^2} y \Big|_0^x dx$$

$$= \int_0^1 x e^{x^2} dx$$

这就能求原函数了

$$= \frac{1}{2} e^{x^2} \Big|_0^1$$

$$= \frac{1}{2} e - \frac{1}{2}$$

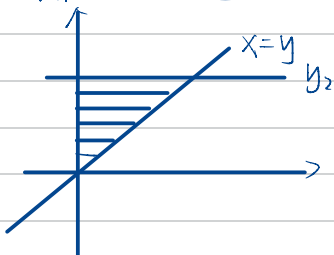
更改下列二次积分的积分次序

$$\textcircled{1} \int_0^1 dy \int_0^y f(x,y) dx$$

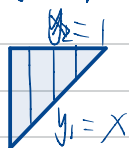
题目

先对 dx 积分

$$x_1=0 \quad x_2=y \quad y_1=0 \quad y_2=1$$



原为横切 改为竖切



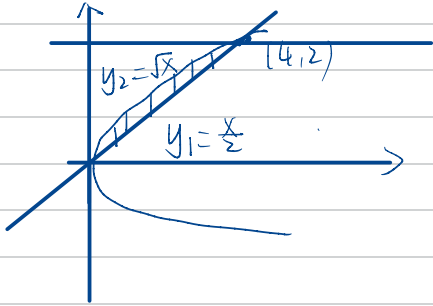
$$\int_0^1 \int_x^1 f(x,y) dy dx$$

$$\int_0^1 dx \int_x^1 f(x,y) dy$$

$$\begin{aligned} x_1 &= y^2 & y_1 &= 0 \\ x_2 &= 2y & y_2 &= 2 \end{aligned}$$

e) $\int_0^2 dy \int_{y^2}^{2y} f(x,y) dx$
横切

$$x_1 = y^2 \quad x_2 = 2y \quad y_1 = 0 \quad y_2 = 2$$



∴ 改竖切

$$\int_0^4 dx \int_{\frac{x}{2}}^{\sqrt{x}} f(x,y) dy$$

$$\int_0^4 dx \int_{\frac{x}{2}}^{\sqrt{x}} f(x,y) dy$$

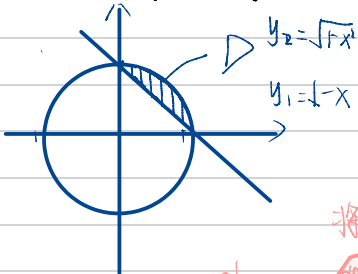
三维空间中 $x=0, y=0, x+y=1, z=0, z=b-(x^2+y^2)$ 求其围成体积
知其平面面积 $\downarrow$ 商 $\uparrow$

$$\iint_D (z_2 - z_1) dx dy$$



$$\therefore \iint_D [b - (x^2 + y^2)] dx dy$$

计算 $\iint_D \frac{x+y}{x^2+y^2} dx dy$ D: $x^2+y^2 \leq 1, x+y \geq 1$



$$\therefore \int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} \frac{x+y}{x^2+y^2} dy$$

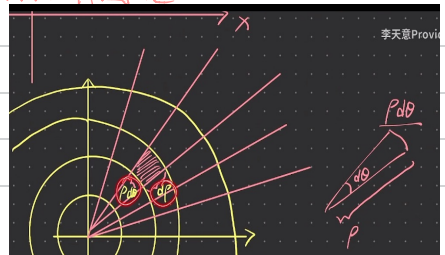
但用这种方法计算困难

∴ 采用极坐标

将 $x = \rho \cos \theta, y = \rho \sin \theta$ 代入

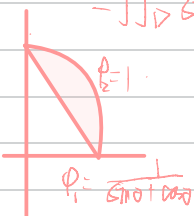
$dx dy = \rho d\rho d\theta$ 弧长 = 半径 $\times$ 圆心角

$$dx dy = \rho d\theta d\rho$$



换为 $\iint_D \frac{\sin\theta + \cos\theta}{\rho} \rho d\rho d\theta$

$= \iint_D \sin\theta + \cos\theta d\rho d\theta$



打出的粒子先穿 ρ_1 再穿 ρ_2

$\int_0^{\frac{\pi}{2}} d\theta \int_{\cos\theta + \sin\theta}^1 \sin\theta + \cos\theta d\rho$

(一般先积 θ 再积 ρ)

从原点射出粒子打中所述区域的角度

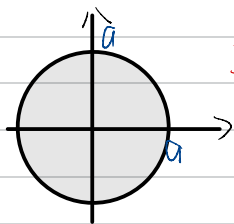
$= \int_0^{\frac{\pi}{2}} (\sin\theta + \cos\theta - 1) d\theta$

$= -\cos\theta + \sin\theta - \theta \Big|_0^{\frac{\pi}{2}}$

$= 1 - \frac{\pi}{2} - (-1)$

$= 2 - \frac{\pi}{2}$

$\iint_D e^{-x^2-y^2} dx dy$, D 是圆心在原点, 半径为 a 的圆



判断是否采用极坐标: ① 积分区域与圆有关

② 被积函数

$$\begin{cases} x^2 + y^2 \Rightarrow \rho^2 \\ \sqrt{x^2 + y^2} \Rightarrow \rho \\ \arctan \frac{y}{x} \Rightarrow \theta \end{cases}$$

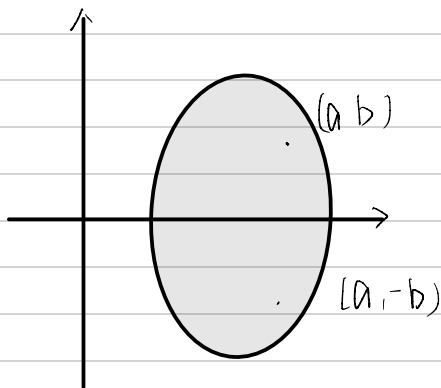
原式 $\iint_D e^{-\rho^2} \rho d\rho d\theta$

$= \int_0^{2\pi} d\theta \int_0^a e^{-\rho^2} \rho d\rho$

$= 2\pi \cdot \left[-\frac{e^{-\rho^2}}{2} \right]_0^a$

$= 2\pi \cdot \frac{(1 - e^{-a^2})}{2}$

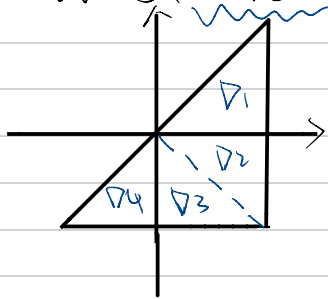
二重积分的对称性



$$\iint_D x^2 y \, dy \, dx \text{ 在椭圆上} = 0$$

在第一个：当 $y = -y$ 时 $x^2 y \Rightarrow -x^2 y$
 $f(x, y) \Rightarrow -f(x, y)$
 $\therefore$ 上下相反抵消

例： $\iint_D y (1 + x e^{\frac{x^2+y^2}{2}}) \, dx \, dy$ 积分区域如图
 (不可能求出原函数，因对称)



划分为 D_1, D_2, D_3, D_4

$D_1 + D_2$ 关于 x 轴对称

$D_3 + D_4$ 关于 y 轴对称

被积： $y + x y e^{\frac{x^2+y^2}{2}}$

在 D_1, D_2 上 $x y e^{\frac{x^2+y^2}{2}}$ 都抵消 (因 y 与 $-y$ 抵消)
 在 D_3, D_4 上 $x y e^{\frac{x^2+y^2}{2}}$ 都抵消 (因 x 与 $-x$ 抵消)

$\therefore$ 只剩积 y

又 D_1, D_2 上 y 可被抵消，可只算 D_3, D_4 上 y

$$\begin{aligned} & \iint_{D_3+D_4} y \, dx \, dy \\ &= 2 \iint_{D_3} y \, dx \, dy = -\frac{2}{3} \end{aligned}$$

轮换对称性

例: $\iint_{D_1} (2x+3y+\sin x) dx dy$
 $\iint_{D_2} (2y+3x+\sin y) dx dy$

$D_1: 2x+3y \leq 5 \quad x \geq 0 \quad y \geq 0$
 $D_2: 2y+3x \leq 5 \quad y \geq 0 \quad x \geq 0$

互相对调, 值不变

例: $I = \iint_D \frac{3\sqrt{\sin x} + 2\sqrt{\sin y}}{\sqrt{\sin^2 x} + \sqrt{\sin^2 y}} dx dy$

$D: x^2 + y^2 \leq 4 \quad x \geq 0 \quad y \geq 0$

① 对调: $\iint_D \frac{3\sqrt{\sin y} + 2\sqrt{\sin x}}{\sqrt{\sin^2 y} + \sqrt{\sin^2 x}} dx dy$

$D: y^2 + x^2 \leq 4 \quad y \geq 0 \quad x \geq 0$

还是没变

② 判断: $D' = D$ (积分区域不变)

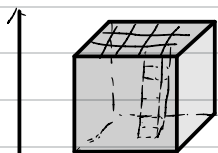
轮换对称性重要条件

③ 相加

$$2I = \iint_D \frac{5\sqrt{\sin x} + 5\sqrt{\sin y}}{\sqrt{\sin^2 x} + \sqrt{\sin^2 y}} dx dy$$

$$= \iint_D 5 dx dy$$

$$I = \iint_D \frac{5}{2} dx dy$$



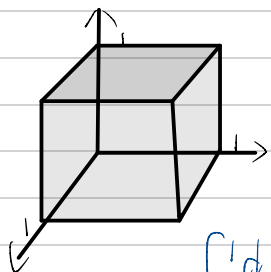
$\rightarrow$ $\begin{matrix} dx \\ dy \\ dz \end{matrix}$ (= 体积元素的长宽, 再横切)

$$\iiint_{\Omega} P(x, y, z) dx dy dz$$

$$\int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} P dz$$

$$\Omega: 0 \leq x \leq 1 \quad 0 \leq y \leq 1 \quad 0 \leq z \leq 1$$

计算 $\iiint_{\Omega} x+y+z \, dx dy dz$



① 投影 $(x, y) \rightarrow$



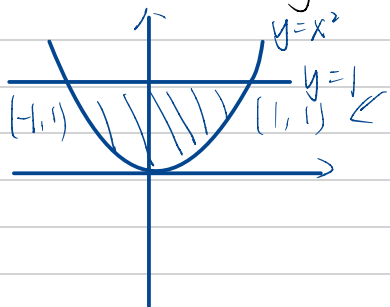
② 上顶、下底
 $z=1 \quad z=0$

$$\int_0^1 dy \int_0^1 dx \int_0^1 (x+y+z) \, dz$$

$$= \int_0^1 dy \int_0^1 (x+y) \frac{1}{2} \, dx$$

$$= \int_0^1 (1+y) dy = y + \frac{y^2}{2} \Big|_0^1 = \frac{3}{2}$$

$$\iiint_{\Omega} xz \, dx dy dz \quad \Omega: z=0 \quad z=y, \quad y=1 \quad y=x^2$$



① 投影

找只跟xy有关面出来

② 上顶: $z=y$ 下底: $z=0$ (找与z有关)

$$\int_{-1}^1 dx \int_{x^2}^1 dy \int_0^y xz \, dz$$

$$\int_{-1}^1 dx \int_{x^2}^1 \frac{xy^2}{2} \, dy$$

$$= \int_{-1}^1 \frac{x-x^3}{6} \, dx$$

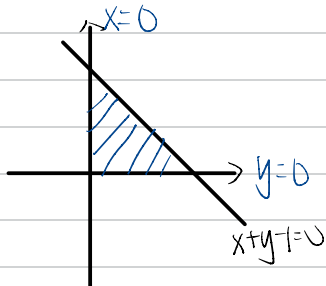
$$= \frac{x^2}{12} - \frac{x^4}{48} \Big|_{-1}^1 = 0$$

例: $\iiint_{\Omega} f(x,y,z) dx dy dz$

$\Omega: z=xy$

$x+y-1=0$

$z=0$



① 投影

消元得

$xy=0$
 $x=0, y=0$

①②与z轴

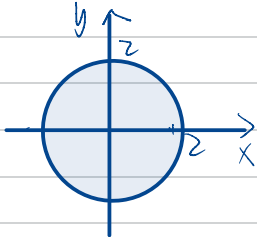
② 上顶: $z_2=xy$ 下底 $z_1=0$

③ $\int_0^1 dx \int_0^{1-x} dy \int_0^{xy} f(x,y,z) dz$

例: $\iiint_{\Omega} z dx dy dz$

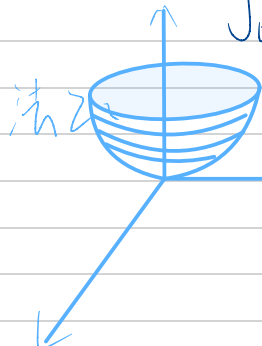
$\Omega: z=x^2+y^2 \quad z=4$

① 投影: 消元得 $x^2+y^2=4$



② 上顶: $z_2=4$ 下底 $z_1=x^2+y^2$

③ $\int_0^{2\pi} d\theta \int_0^2 \rho d\rho \int_{\rho^2}^4 z dz$ (圆, 用极坐标)



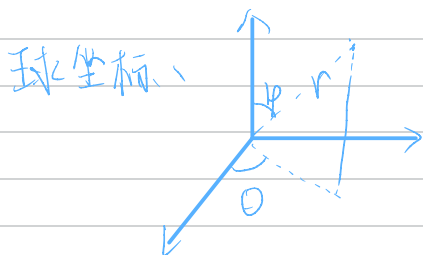
(密度不会随上下变化) z均匀分布



$x^2+y^2 \cdot z = r$
 $\therefore r = \sqrt{z}$

$V = \pi r^2 \cdot dz$
密度 z

全部 $m = \int_0^4 \pi z^2 dz$



$$z = r \cos \varphi$$

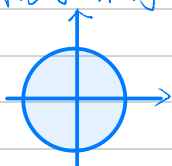
$$x = r \sin \varphi \cos \theta$$

$$y = r \sin \varphi \sin \theta$$

例: $\iiint_{\Omega} z \, dx \, dy \, dz$ $\Omega: x^2 + y^2 + (z-a)^2 \leq a^2$, $x^2 + y^2 \leq z^2$ $z^2 + (z-a)^2 = a^2$

消元得 $x^2 + y^2 = a^2$

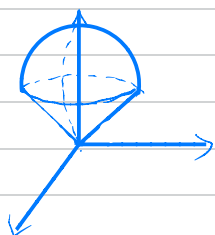
投影:



$$z^2 = 2az - x^2 - y^2$$

上顶: $z_1 = a + \sqrt{a^2 - x^2 - y^2}$

下底: $z_2 = \sqrt{x^2 + y^2}$



$$z \Rightarrow r \cos \varphi$$

$$dx \, dy \, dz \Rightarrow r^2 \sin \varphi \, dr \, d\theta \, d\varphi$$

$$= \iiint_{\Omega} r^3 \cos \varphi \sin \varphi \, dr \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2a \cos \varphi} r^3 \cos \varphi \sin \varphi \, dr$$

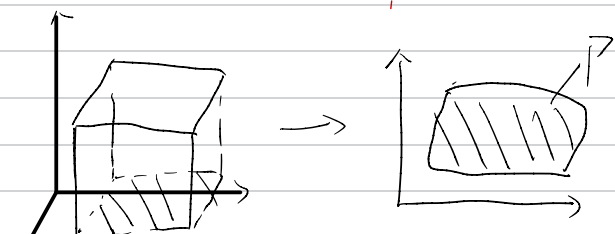
$$x^2 + y^2 + (z-a)^2 = a^2$$

$$\begin{array}{l} \downarrow \begin{array}{l} r \\ \theta \\ \varphi \end{array} \quad \begin{array}{l} x = r \sin \varphi \cos \theta \\ y = r \sin \varphi \sin \theta \\ z = r \cos \varphi \end{array} \end{array}$$

$$r = 2a \cos \varphi$$

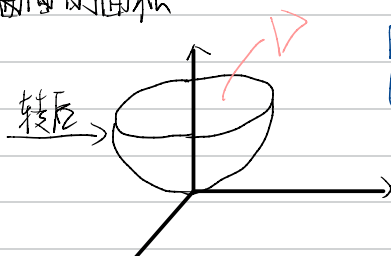
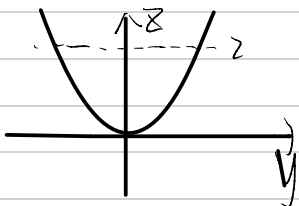
重积分应用

- 空间曲面面积: $A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$
- 平面区域 D 绕直线旋转的体积: $2\pi \iint_D \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}} dx dy$
- 转动惯量 $\iint_D \rho r^2 dx dy$



$$A = \iint_D \sqrt{1 + (f'_x)^2 + (f'_y)^2} dx dy$$

例: 在 yOz 平面上抛物线 $z = y^2$ 从 $z=0$ 到 $z=2$ 的一段绕 z 轴旋转一周形成曲面的面积



所得曲面方程:
(z 不动, $y = \pm \sqrt{x^2 + y^2}$ 代入)

$$z = x^2 + y^2$$

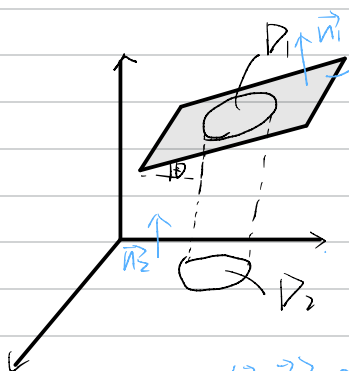
$$f'_x = \frac{\partial z}{\partial x} = 2x \quad f'_y = \frac{\partial z}{\partial y} = 2y$$

$$\iint_D \sqrt{1 + 4x^2 + 4y^2} dx dy$$

所求区域: 投影的面积



$$\begin{aligned} &= \iint_D \sqrt{1 + 4\rho^2} \rho d\rho d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \sqrt{1 + 4\rho^2} \rho d\rho \end{aligned}$$



$$Ax + By + Cz + D = 0$$

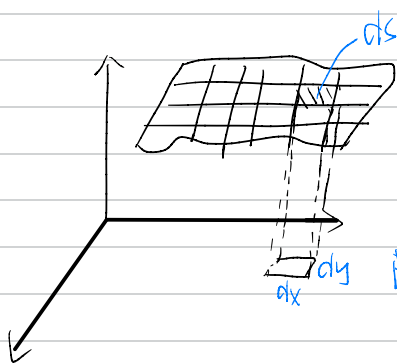
$$D_1 \cos \theta = D_2$$

$$\vec{n}_1 = (A \ B \ 0)$$

$$\vec{n}_2 = (0 \ 0 \ 1)$$

$$\cos \angle(\vec{n}_1, \vec{n}_2) = \cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{C}{\sqrt{A^2 + B^2 + C^2}}$$

$$\therefore D_1 = \frac{\sqrt{A^2 + B^2 + C^2}}{C} D_2$$



$$\therefore ds = \frac{\sqrt{A^2 + B^2 + C^2}}{C} dx dy = \sqrt{f_x^2 + f_y^2 + 1} dx dy$$

证明:

$z = f(x, y)$ 在 (x_0, y_0, z_0) 处切平面:

$$\frac{f'_x}{A}(x - x_0) + \frac{f'_y}{B}(y - y_0) + \frac{f'_z}{C}(z - z_0) = 0$$

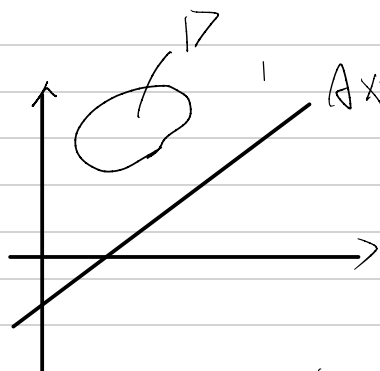
$$F = f(x, y) - z$$

$$F'_x = f'_x$$

$$F'_y = f'_y$$

$$F'_z = -1$$

忙

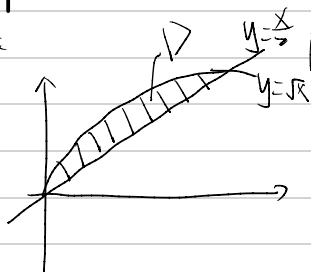


$$Ax + By + C = 0$$

求该直线旋转后的体积

$$V = 2\pi \iint_D \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}} dx dy$$

例:



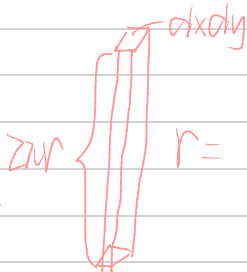
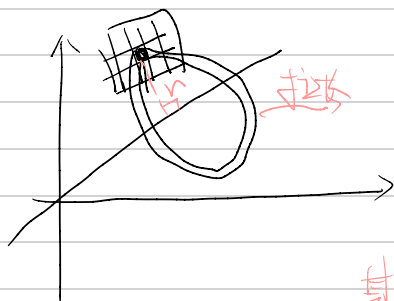
绕着 $\frac{x}{2}$ 转一周后形成的体积

$$V = 2\pi \iint_D \frac{|x - 2y|}{\sqrt{1^2 + 2^2}} dx dy$$

$$= \frac{2\sqrt{5}\pi}{5} \int_0^4 dx \int_{\frac{x}{2}}^{\sqrt{x}} |2y - x| dy$$

$$= \frac{16\sqrt{5}}{75}$$

公式 $V = 2\pi \iint_D \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}} dx dy$ 推导:

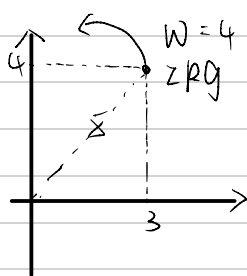


$$r = \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}}$$

其中一块的体积: $2\pi \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}} dx dy$

∴ 全部一起积: $2\pi \iint_D \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}} dx dy$

转动惯量

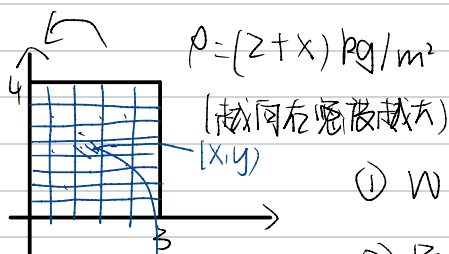


① $V =$

② 动能 =

① $V = \omega r = 5 \times 4 = 20 \text{ m/s}$

② $\frac{mV^2}{2} = \frac{2 \times 20^2}{2} = 400$



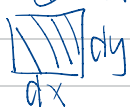
① $\omega = 4 \text{ rad/s}$ 具有多少动能

② 角速度为 ω 时, 动能为?

(速度不均
密度不均)

利用微积分

其中一个格子



面积: $dx dy$

质量: $(2+x) dx dy$

速度: $V = \omega r$
 $= 4 \cdot \sqrt{x^2 + y^2}$

动能: $\frac{mV^2}{2} = \frac{(2+x) dx dy \cdot 16(x^2 + y^2)}{2} = 8(x^2 + y^2)(2+x) dx dy$

∴ 全部的格子: $\int_0^3 dx \int_0^4 8(x^2 + y^2)(2+x) dy$

$= 3016 \text{ J}$

假设 ρ 也未知, $\rho = \rho(x, y)$ w 也未知

面积: $dx dy$

质量: $\rho(x, y) dx dy$

速度: $v = wr$
 $= w\sqrt{x^2 + y^2}$

动能: $\frac{mv^2}{2} = \frac{\rho(x, y) dx dy \cdot w^2 (x^2 + y^2)}{2}$

整体动能: $\iint_D \frac{\rho(x, y) dx dy \cdot w^2 (x^2 + y^2)}{2}$

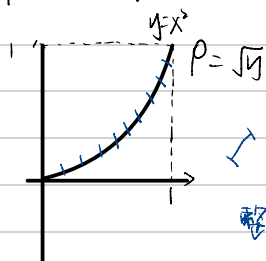
$$= \frac{w^2}{2} \iint_D \rho(x, y) (x^2 + y^2) dx dy$$

转动惯量

$$\therefore \text{即求 } I = \iint_D \rho \underbrace{r^2}_{\substack{\downarrow \\ \text{点到转轴的距离}}} dx dy$$

点到转轴的距离

第一类曲线积分



一小段的质量: ρds

整段质量: $\int_L \rho ds$

$$= \int_L \sqrt{y} ds$$

$$= \int_L \sqrt{y} \sqrt{(dx)^2 + (dy)^2}$$

(目标: 用一个变量表示全部)

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$(\sqrt{(dx)^2 + (dy)^2})^2 = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (2x)^2} dx$$

y 对 x 求导

题中 $y = x^2$

$$= \int_L x \cdot \sqrt{1 + 4x^2} dx$$

$$= \int_0^1 x \sqrt{1 + 4x^2} dx$$

$$= \frac{1}{8} \int_0^1 \sqrt{1 + 4x^2} d(1 + 4x^2)$$

$$= \frac{\sqrt{5} - 1}{12}$$

$$\int_L \rho(x, y) ds \quad (\text{换})$$

用 $\sqrt{(dx)^2 + (dy)^2}$

例: $\int_L (x + 2y) ds$, L 是连接 $(2, 0)$ $(0, 1)$ 线段

换 x :

由题求 $y = -\frac{1}{2}x + 1$

$$\int_L (x + 2 - x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_L 2 \sqrt{1 + \frac{1}{4}} dx$$

$$= \int_L \sqrt{5} dx$$

$$= \sqrt{5} \times \frac{1}{2} = \frac{\sqrt{5}}{2}$$

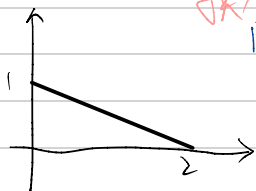
换 y :

$$\int_L (2 - 2y + 2y) \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$$

$$= \int_L 2 \sqrt{5} dy$$

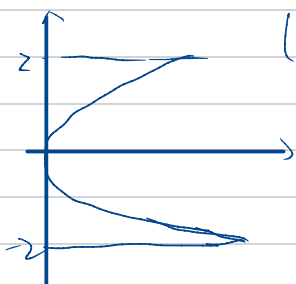
$$= \int_0^1 2\sqrt{5} dy$$

$$= 2\sqrt{5}$$



设 $L: x=y^2 \ (-2 \leq y \leq 2)$ $\int_L f(x,y) ds$

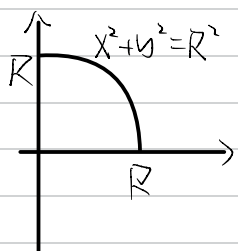
① 转化为关于 x 积分 ② 转化为 y 积分



(按 $x: y = \pm \sqrt{x}$: 分类讨论)

$$\textcircled{1} \int_{L_1} f(x,y) ds + \int_{L_2} f(x,y) ds \\ = \int_0^4 f(x, \sqrt{x}) \sqrt{1+\frac{1}{4x}} dx + \int_0^4 f(x, -\sqrt{x}) \sqrt{1+\frac{1}{4x}} dx$$

$$\textcircled{2} \int_L f(y^2, y) ds \\ = \int_{-2}^2 f(y^2, y) \sqrt{1+(2y)^2} dy \\ = \int_{-2}^2 f(y^2, y) \sqrt{1+4y^2} dy$$



$$\int_L f(x^2+y^2) ds$$

$$\text{老做法: } \int_L R^2 \sqrt{(dx)^2 + (dy)^2}$$

$$\text{又 } \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

↓
y 对 x 求导麻烦

新出路: 不用 x 或 y 替换, 用其他量替换

$$\sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$



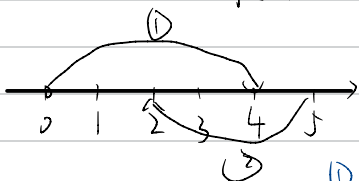
$$\begin{cases} x = R \cos \theta \\ y = R \sin \theta \end{cases}$$

$$\begin{aligned} \frac{dx}{d\theta} &= -R \sin \theta \\ \frac{dy}{d\theta} &= R \cos \theta \end{aligned}$$

$$\therefore \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\ = R d\theta$$

第二类曲线积分

在 x 轴上, $F=2x$ $\begin{cases} \textcircled{1} \text{ 质点从 } x=1 \text{ 移动到 } x=4 \\ \textcircled{2} \text{ 从 } x=5 \text{ 移动到 } x=2 \end{cases}$



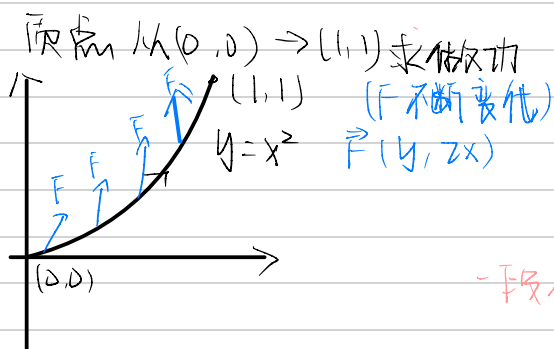
$\therefore F$ 随 x 不断变化

$\therefore$ 切分为 dx

① 在一段内: $Fdx = 2x dx$

整段: $\int_1^4 F dx = \int_1^4 2x dx = x^2 \Big|_1^4 = 15$

② $\int_5^2 F dx = \int_5^2 2x dx = x^2 \Big|_5^2 = -21$
(注意上下限, 是从 $5 \rightarrow 2$)



(分解 $\vec{F}$ 为 dx 和 dy)

一段做功: $F_x \cdot dx + F_y \cdot dy$
 $= y dx + 2x dy$

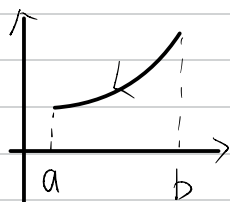
整段做功: $\int_L y dx + 2x dy$
 $\downarrow y = x^2$
 $\int_L x^2 dx + 2x d(x^2)$
 $= \int_L x^2 dx + 4x^2 dx$
 $= \int_L 5x^2 dx$
 $= \frac{5}{3} x^3 \Big|_0^1 = \frac{5}{3}$

第一类曲线积分: $\int_L f(x,y) ds$

第二类 $\int_C P dx + Q dy$

有方向 (从 $\rightarrow$)

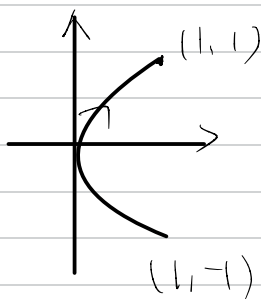
PS: 第一类不关注上下限的方向



$$= \int_a^b \dots dx$$

$$= \int_b^a \dots dx$$

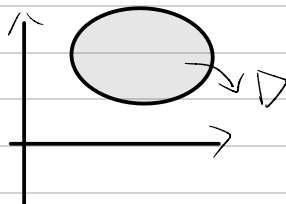
例: $\int_L xy dx$, L 是 $x = y^2$ 从 $(1, -1)$ 到 $(1, 1)$ 的一段曲线
第二类 (没有 ds)



$$\begin{aligned} & \text{换 } y \\ & \int_L y^3 d(y^2) \\ &= \int_C 2y^4 dy \\ &= \frac{2}{5} y^5 \Big|_{-1}^1 \\ &= \frac{4}{5} \end{aligned}$$

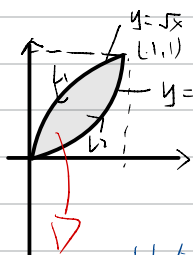
格林公式

正向边界: 区域在左手边



逆向边界: 区域在右手边

$$\begin{aligned} & \int_C P dx + Q dy \\ &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \end{aligned}$$



求 $\int_{L_1+L_2} (2xy - x^2) dx + (x + y^2) dy$

① 换成 x

(L_1 与 L_2 未述
或不同分类
讨论)

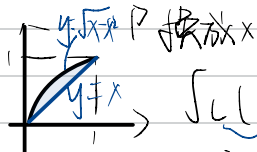
$$\begin{aligned} & \int_{L_1} (2x\sqrt{x} - x^2) dx + (x + x) d\sqrt{x} \\ & + \int_{L_2} (2xx^2 - x^2) dx + (x + x^4) d(x^2) \\ & = \int_1^0 (2x\sqrt{x} - x^2) dx + 2x dx \\ & + \int_0^1 (2x^3 - x^2) dx + (x + x^4) dx^2 \end{aligned}$$

用格林公式:

$$\begin{aligned} & \int_{L_1+L_2} \underbrace{(2xy - x^2)}_P dx + \underbrace{(x + y^2)}_Q dy \\ & = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ & = \iint_D (1 - 2x) dx dy \\ & = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (1 - 2x) dy \\ & = \frac{1}{30} \end{aligned}$$

定理: $\int_C P dx + Q dy$ 如果有 $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ 这个曲线积分与路径无关

例: $\int_L (x^2 - y) dx - (x + \sin^2 y) dy$, L 是 $y = \sqrt{2x - x^2}$ 从 $(0,0)$ 到 $(1,1)$ 的一段

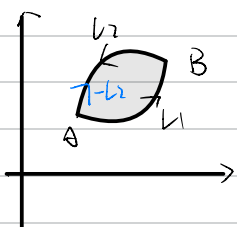


$\int_L (x^2 - \sqrt{2x - x^2}) dx + (x + \sin^2(\sqrt{2x - x^2})) d\sqrt{2x - x^2}$

$$\frac{\partial Q}{\partial x} = -1 \quad \frac{\partial P}{\partial y} = -1$$

$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \therefore$ 与路径无关

改成 $y = x$ 代入后
简单好算



$$\begin{aligned} & \int_{l_1} P dx + Q dy + \int_{l_2} P dx + Q dy \\ &= \int_{l_1+l_2} P dx + Q dy \\ &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \end{aligned}$$

(封闭图形用格林公式)

若 $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

则 $\int_{l_1} P dx + Q dy + \int_{l_2} P dx + Q dy = 0$

$$\int_{l_1} P dx + Q dy = - \int_{l_2} P dx + Q dy$$

$$= \int_{-l_2} P dx + Q dy$$

($-l_2$ 即方向改变)

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ 即与路径无关, 即证

$\oint_L F_x dx + F_y dy$

$$\begin{aligned} & (F_{yL} - F_{yR}) dy - (F_{xL} - F_{xR}) dx \\ &= \int \left[\frac{F_{yL} - F_{yR}}{dx} - \frac{F_{xL} - F_{xR}}{dy} \right] dx dy \\ &= \iint \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right] dx dy \end{aligned}$$

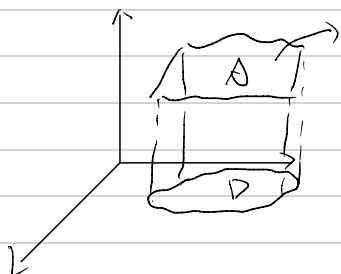


格林公式

第一类曲面积分

$$\iint_{\Sigma} \rho(x, y, z) ds$$

Σ : 代表曲面 (无方向)

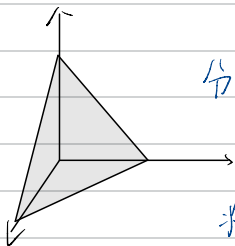


原来 $A = \iint_D \sqrt{1 + (f'_x)^2 + (f'_y)^2} dx dy$

$\therefore M = \underbrace{\rho(x, y, z)}_{\text{密度}} \cdot \underbrace{\iint_D \sqrt{1 + (f'_x)^2 + (f'_y)^2} dx dy}_{\text{面积}}$

↓
现在的 ds

例: $\iint_{\Sigma} xyz ds$, $\Sigma: x+y+z=1$ 被 $x=0, y=0, z=0$ 截取



R 两投影区域

分为二类: ① 只含 x, y

② 含有 z 的

将 $z=0$ 代入: $x+y=1$

$$\iint_{\Sigma} xyz ds \xrightarrow{z=1-x-y} \iint_{\Sigma} xy(1-x-y) ds$$

$$ds = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$$

$$= \sqrt{3} dx dy$$

$$\begin{aligned} \therefore \iint_D xy(1-x-y) \sqrt{3} dx dy \\ &= \int_0^1 dx \int_0^{1-x} \sqrt{3} (1-x-y) dy \\ &= \frac{\sqrt{3}}{12} \end{aligned}$$

第二类曲面积分

$$\iint_{\Sigma} F_x dy dz + F_y dx dz + F_z dx dy$$

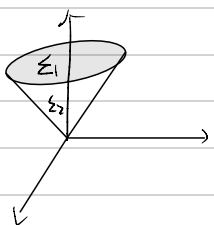
$F_x(x, y, z)$ Σ : 代表曲面 (有方向)

$F_y(x, y, z)$

$F_z(x, y, z)$

例: $\iint_{\Sigma} (x^2+y^2) ds$ Σ : 是由 $z=\sqrt{x^2+y^2}$, $z=1$ 围成的空间立体区域的整个表面

$$\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy \longrightarrow$$



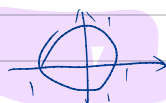
$$\text{原式: } \iint_{\Sigma_1} (x^2+y^2) ds + \iint_{\Sigma_2} (x^2+y^2) ds$$

$$\textcircled{1} \iint_{D_1} x^2+y^2 \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy \quad (z=1)$$

$$= \iint_{D_1} x^2+y^2 dx dy$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho^3 d\rho = \frac{\pi}{2}$$

D_1 (投影)



$$\textcircled{2} \iint_{D_2} x^2+y^2 \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy \quad (z=\sqrt{x^2+y^2})$$

$$= \iint_{D_2} x^2+y^2 \sqrt{1 + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2} dx dy$$

$$= \iint_{D_2} (x^2+y^2) \sqrt{2} dx dy$$

$$= \frac{\sqrt{2}}{2} \pi$$

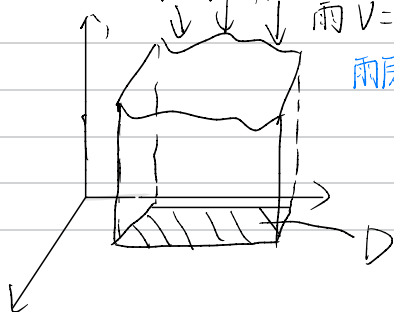
$(D_1, D_2 \text{ 在 } xy \text{ 上的投影面积相同})$

$\therefore$ 共为 $\frac{\sqrt{2}+1}{2} \pi$

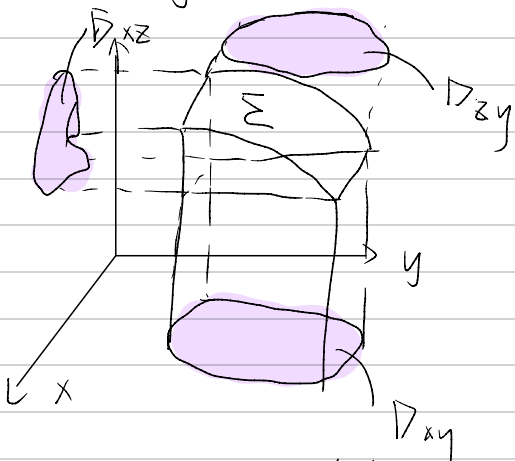
第二类曲面积分

雨 $V=f(x,y,z)$

雨雨流量大小只与其投影面积有关, 与平面面积无关



$$\iint_{\Sigma} P dy dz + Q dz dx + R dx dy$$

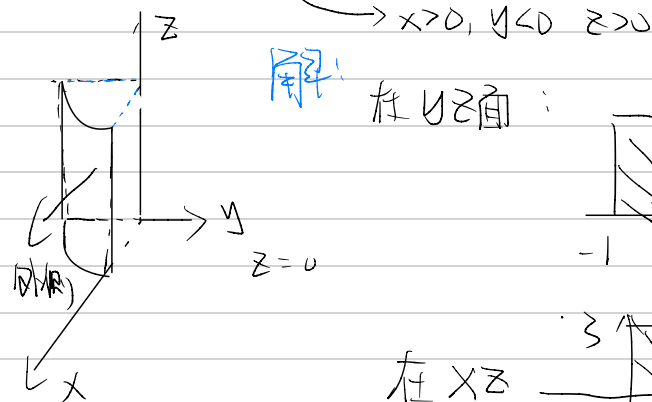


记为 Σ 在三个平面上的投影

$$\therefore P = D_{yz} \quad Q = D_{xz} \quad R = D_{xy}$$

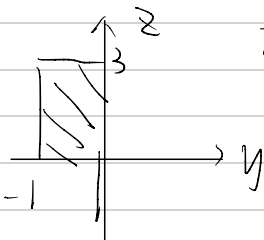
三个部分

例: $\iint_{\Sigma} (x dy dz + y dz dx + z dx dy)$ Σ : 柱面 $x^2 + y^2 = 1$ 被 $z=3, z=0$ 截取
后在第四卦限的外侧区域

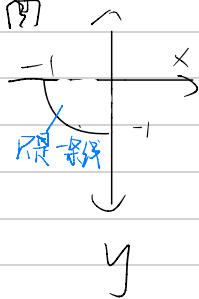


解:

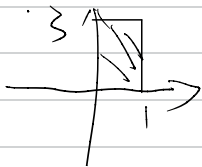
在 yz 面:



在 xy 面:



在 xz 面:



yz 面:

$$\iint_{\Sigma} x dy dz = \iint_{\Sigma} yz \sqrt{1-y^2} dy dz$$

(考虑在 yz 面
投影为沿 x 轴
x 轴与 y 轴不
同, 锐角)

$$= \int_{-1}^0 dy \int_0^3 \sqrt{1-y^2} dz = \frac{3}{4} z$$

xz 面:

$$\iint_{\Sigma} y dz dx = \int_0^1 dx \int_0^3 -\sqrt{1-x^2} dz$$

$= -\frac{3}{4} z$ (与 xz 面投影, (y 轴) 成钝角加负) $\therefore$ 为 $\frac{3}{4} z$

$\therefore$ 三个相加

$$\frac{3}{2} z$$

无穷级数

$$\left\{ \begin{array}{l} \text{常数项级数: } a_1 + a_2 + a_3 + \dots \\ \text{幂级数: } a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\ \text{傅里叶级数: } a_0 + a_1 \cos x + a_2 \cos 2x + \dots \\ \quad + b_1 \sin x + b_2 \sin 2x + \dots \end{array} \right. \left. \begin{array}{l} \text{正项} \\ \text{交错} \end{array} \right.$$

1. 等比级数

$$(1) 1 + 2 + 4 + 8 + 16 + \dots = \infty = \sum_{n=0}^{\infty} 2^n \quad \downarrow \text{无限大}$$

$$(2) \underbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots}_{\text{无限项}} = \frac{a_1(1-q^n)}{1-q} = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = \frac{2}{1 - \frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4$$

发散
收敛

收敛条件: $|q| < 1$

2. 调和级数:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty \quad \text{发散}$$

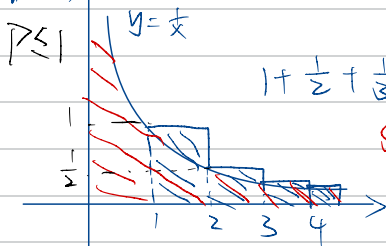
3. P级数

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\left\{ \begin{array}{l} p > 1 \quad \text{收敛} \\ p \leq 1 \quad \text{发散} \end{array} \right.$$

(若 $p=1$ 则为调和级数)

证: 调和级数



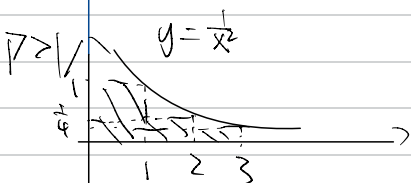
$$1 + \frac{1}{2} + \frac{1}{3} + \dots = S_{\text{红}}$$

$$S_{\text{红}} = \int_1^{\infty} \frac{1}{x} dx = \ln x \Big|_1^{\infty} = \infty$$

$$S_{\text{蓝}} > S_{\text{红}}$$

且 $S_{\text{红}}$ 已无限

$\therefore S_{\text{蓝}}$ 无限发散即调和级数发散



同理 $S_{\text{蓝}} < S_{\text{红}}$

$$S_{\text{红}} = \int_1^{\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^{\infty} = 1$$

且 $S_{\text{红}} \text{ max} = 1 \therefore S_{\text{蓝}} \text{ 收敛}$

$$\sum_{n=1}^{\infty} 3^n \sin \frac{z}{4^n} = 3 \cdot \sin \frac{z}{4} + 9 \cdot \sin \frac{z}{16} + \dots$$

① 根审敛

② 比值审敛 ~~★~~

③ 比较审敛 } 被编

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \dots$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$$

$\rho > 1$ — 发散
 $\rho < 1$ — 收敛
 $\rho = 1$ — 方法失败

收敛:

$$\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n \text{ (比较)}$$

若 $b_n \geq a_n$ 且 b_n 收敛
 则 $\sum a_n$ 收敛

同理

若 $b_n \leq a_n$ 且 b_n 发散
 则 $\sum a_n$ 发散

用上述例题: ②:

$$a_n = 3^n \sin \frac{z}{4^n}$$

$$a_{n+1} = 3^{n+1} \sin \frac{z}{4^{n+1}}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{3^{n+1} \sin \frac{z}{4^{n+1}}}{3^n \sin \frac{z}{4^n}}$$

$$= \frac{3 \frac{z}{4^{n+1}}}{\frac{z}{4^n}} \quad (\text{等价无穷小})$$

$$= \frac{3}{4} < 1$$

$\therefore \rho < 1 \therefore$ 收敛

极限:

$$\sum_{n=0}^{\infty} a_n \text{ 如果有}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$$

$C \in (0, +\infty)$

$$\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} b_n$$

只有相收敛

③ 放缩

$$a > 0, \sin a < a$$

$$\therefore \sin \frac{z}{4^n} < \frac{z}{4^n}$$

$$\therefore \text{找个大数, } \sum 3^n \sin \frac{z}{4^n} < \sum 3^n \frac{z}{4^n}$$

$$\text{即 } a = \frac{3}{4} < 1$$

$\therefore$ 收敛

则原式也收敛

③ 极限

例: $\sum_{n=1}^{\infty} \frac{1}{n!}$

$n \rightarrow \infty \quad 3^n \sin \frac{2\pi}{3^n} \sim 3^n \frac{2\pi}{3^n}$

解: ② $a_n = \frac{1}{n!}$

$a_{n+1} = \frac{1}{(n+1)!}$

$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} = 0 < 1$

$\rho < 1 \therefore$ 收敛

$\lim_{n \rightarrow \infty} \frac{3^n \sin \frac{2\pi}{3^n}}{3^n \frac{2\pi}{3^n}} = 1$

于是 $\sum_{n=1}^{\infty} 3^n \frac{2\pi}{3^n}$ 与原式有相同敛散性

例: $\sum_{n=1}^{\infty} \sqrt{n+1} (1 - \cos \frac{1}{n})$

解: ② 用比值审敛

$a_n = \sqrt{n+1} (1 - \cos \frac{1}{n})$

$a_{n+1} = \sqrt{n+2} (1 - \cos \frac{1}{n+1})$

$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{1 - \cos \frac{1}{n+1}}{1 - \cos \frac{1}{n}}$

$= \lim_{n \rightarrow \infty} \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{\frac{1}{2} (\frac{1}{n+1})^2}{\frac{1}{2} (\frac{1}{n})^2}$

$= \lim_{n \rightarrow \infty} \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{n^2}{(n+1)^2}$

$= \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{2}{n}}}{\sqrt{1+\frac{1}{n}}} \cdot \frac{1}{(1+\frac{1}{n})^2} = 1$

$\rho = 1 \therefore$ 该方法不可用

③ 用极限法

$\sum_{n=1}^{\infty} \sqrt{n+1} (1 - \cos \frac{1}{n})$

相同敛散性

$\frac{1}{\sqrt{n}} \cdot \frac{1}{2n^2} = \frac{1}{2n^{\frac{5}{2}}}$

p 级数 $p > 1$ 收敛

例: $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

② 比值: $\lim_{n \rightarrow \infty} \frac{\frac{1}{\ln(n+2)}}{\frac{1}{\ln(n+1)}} = \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln(n+2)}$

$\lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln(n+2)} = 1$

$\rho = 1$ 不能用

③ 比较—放缩 ($\ln(n+2) < n$)

$\therefore x > 0 \quad \ln^{+x} < x$

$\therefore \frac{1}{\ln^{+x}} > \frac{1}{x}$

$\frac{1}{x}$ 发散

$\therefore \frac{1}{\ln^{+x}}$ 为大 n 也发散

交错级数

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

① 通项绝对值递减

② 绝对值趋于0

交错级数 $\left\{ \begin{array}{l} \text{条件收敛 (改条件后不收敛)} \\ \text{绝对收敛 (改条件后也收敛)} \end{array} \right.$

① $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$

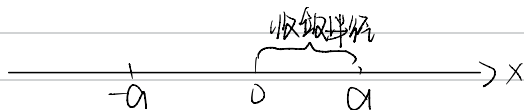
将“-” $\rightarrow$ “+”后

为级数, $p = \frac{1}{2} < 1$ 发散
 $\therefore$ 为条件收敛

幂级数

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

例: $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$



阿贝尔定理: 则 $x=0$ 时收敛, 则找到 $-a$, $(-a, a)$ 范围内有收敛
 发散, 则找到 $-a$, $(-\infty, -a)$ (a, ∞) 发散

(1) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ 求收敛半径

x^n 前面的系数 $a_n = \frac{(-1)^{n+1}}{n}$

$\therefore a_{n+1} = \frac{(-1)^{n+2}}{n+1}$

$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$

$r = \frac{1}{\rho} = 1$ $\therefore -1 < x < 1$ 收敛. (± 1 单独拿出来讨论)

收敛半径

例: $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

x^n 系数 $a_n = \frac{1}{n!}$

$a_{n+1} = \frac{1}{(n+1)!}$

$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = 0$ (极限情况)

$\rho = \frac{1}{\rho_n} = +\infty$
 $\rho_n \nearrow$ 趋近于 0

$\therefore$ 全范围收敛

$1 + x + 2!x^2 + 3!x^3 + \dots$

$a_n = n!$

$a_{n+1} = (n+1)!$

$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} n+1 = +\infty$

$\rho = \frac{1}{\rho_n} = 0$
 $\rho_n \nearrow$ 无穷

$\therefore$ 全范围不收敛即全发散

$\lim_{n \rightarrow \infty} \frac{x^{2n+1}}{2n+1}$

PS: a_n 为 x^n 的系数

别代错为 x^{2n+1} 的系数

(碰到缺项的幂级数用此方法)

$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots + \frac{x^{2n+1}}{2n+1} + \frac{x^{2n+3}}{2n+3}$

(只有奇数项的数)

按照常数项比值收敛

$\lim_{n \rightarrow \infty} \frac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}} < 1$ 求 x 范围

$= \lim_{n \rightarrow \infty} x^2 \frac{2n+1}{2n+3}$

$\Rightarrow$ 若要收敛 $-1 < x < 1$

验证 $x=1: 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$
 ... 发散
 (调和级数)
 $x=-1: 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$
 ... 收敛

$$f(x) = f(x_0) + \frac{(x-x_0)f'(x_0)}{1!} + \frac{(x-x_0)^2 f''(x_0)}{2!} + \dots$$

$$\left. \begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned} \right\} -\infty < x < +\infty$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad -1 < x < 1$$

$$\text{eg: } \frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \dots \quad -1 < 2x < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad -1 < x < 1$$

$$\text{两边} \int \frac{1}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad -1 < x < 1$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \quad -1 < x < 1$$

$$\text{两边} \int \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad -1 < x < 1$$

例: (1) 将函数展开为幂级数

① $\frac{1}{2-x}$ (像 $\frac{1}{1-x}$, 改造为 $\frac{1}{1-\frac{x}{2}}$)

$$\begin{aligned} \frac{1}{2-x} &= \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} \\ &= \frac{1}{2} \left(1 + \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \dots \right) \\ &= \end{aligned}$$

例：将 $\sin x$ 展开成 $(x - \frac{\pi}{4})$ 的幂级数

$$(\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots)$$

令 $t = x - \frac{\pi}{4}$, $x = t + \frac{\pi}{4}$ (换元)

$$\sin x = \sin(t + \frac{\pi}{4})$$

$$= \frac{\sqrt{2}}{2} (\sin t + \cos t)$$

$$= \frac{\sqrt{2}}{2} (t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \dots + 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \dots)$$

$$\sin x = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} [(-1)^n \frac{t^{2n+1}}{(2n+1)!} + (-1)^n \frac{t^{2n}}{(2n)!}]$$

$\downarrow t = x - \frac{\pi}{4}$

$$\sin x = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} [(-1)^n \frac{(x - \frac{\pi}{4})^{2n+1}}{(2n+1)!} + (-1)^n \frac{(x - \frac{\pi}{4})^{2n}}{(2n)!}]$$

例：(1) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$ (e^x 展开式)

联想到 e^x 的展开

$$(2) \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$$

(差比数列)

联想到 $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

(但 $-1 < x < 1$)

$$(3) \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

求导后

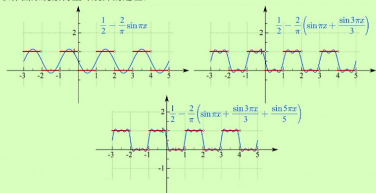
联想到 $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$\therefore$ 原式 $\int \frac{1}{1-x} dx = -\ln|1-x| + C$

傅里叶级数

根据收敛定理, $f(x)$ 在 $x=1$ 处间断, 且左右极限的平均数为 0.5, 所以在 $x=1$ 处傅里叶级数之和为 0.5;
 $f(x)$ 在 $x=1.5$ 处连续, 且取值为 1, 所以在 $x=1.5$ 处傅里叶级数之和为 1.

我们可以作图来观察傅里叶展开的过程:



如果一个周期为 $2L$ 的周期函数 $f(x)$:

(1) 在一个周期内连续或者只有有限个第一类间断点.

(2) 在一个周期内最多有有限个极值点.

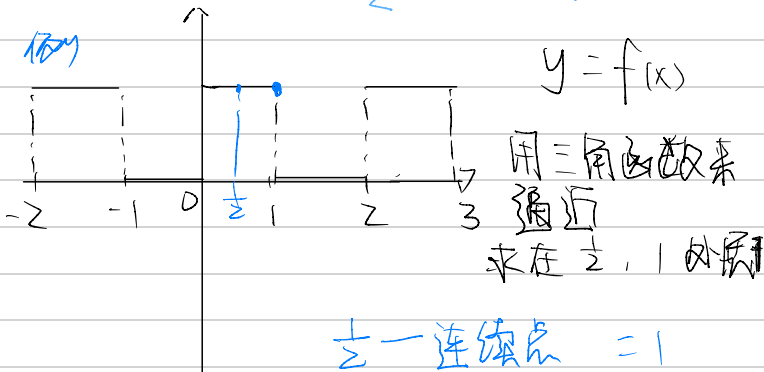
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

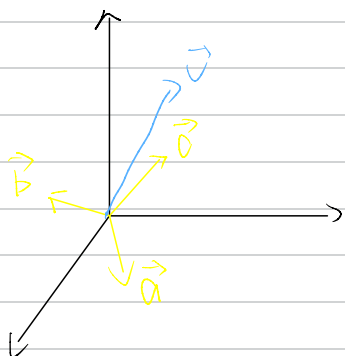
常考: 收敛定理: 傅里叶级数收敛, 取值 S 特点:

$$S(x) = \begin{cases} f(x), & x \text{ 为 } f(x) \text{ 连续点} \\ \frac{f(x) + f(x^+)}{2}, & x \text{ 为 } f(x) \text{ 间断点} \end{cases}$$



$$\frac{1}{2} \text{ — 连续点} = 1$$

$$1 \text{ — 间断点} = \frac{f(1^-) + f(1^+)}{2} = \frac{1}{2}$$



$$\vec{V} = m\vec{a} + n\vec{b} + p\vec{c}$$

$$m = \frac{\vec{a} \cdot \vec{V}}{\vec{a} \cdot \vec{a}}$$

$$n = \frac{\vec{b} \cdot \vec{V}}{\vec{b} \cdot \vec{b}}$$

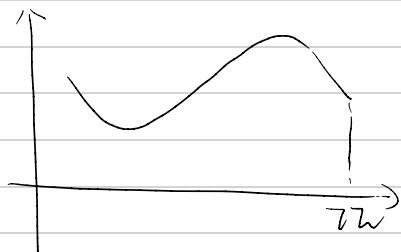
$$p = \frac{\vec{c} \cdot \vec{V}}{\vec{c} \cdot \vec{c}}$$

∞维 $(f(x_1), f(x_2), f(x_3), f(x_4), \dots)$
(函数)

$$\begin{matrix} \cos x & \cos 2x & \cos 3x & \cos 4x \\ \sin x & \sin 2x & \sin 3x & \sin 4x \end{matrix} \quad (0 \sim 2\pi)$$

上下化简法，两两相乘 = 0
与自身相乘 = π

$$\int \sin x \cos x dx = 0$$



$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x \\ + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

$$a_1 = \frac{f(x) \cos x}{\cos x \cos x} = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x dx$$

复变函数

1. 欧拉公式: $e^{i\theta} = \cos\theta + i\sin\theta$
 $r e^{i\theta} = r(\cos\theta + i\sin\theta)$

特殊形式 $\theta=0$, $e^{0i} = \cos 0 + i\sin 0 = 1$

$\theta = \frac{\pi}{2}$ $e^{\frac{\pi}{2}i} = \cos \frac{\pi}{2} + i\sin \frac{\pi}{2} = i$

☆ $\theta = \pi$ $e^{\pi i} = \cos \pi + i\sin \pi = -1$ ($e^{\pi i} + 1 = 0$)

$\theta = -\frac{\pi}{2}$ $e^{-\frac{\pi}{2}i} = \cos -\frac{\pi}{2} + i\sin -\frac{\pi}{2} = -i$

(也牢记逆推)

2. 复平面

$$3+4i \rightarrow (3, 4)$$

$$\frac{x}{y} \rightarrow (x, y)$$

共轭复数 $\bar{z}$

结论: ① $z + \bar{z} = z \times \text{实部}$

② $z - \bar{z} = z \times \text{虚部}$

③ $z \cdot \bar{z} = |z|^2$

④ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

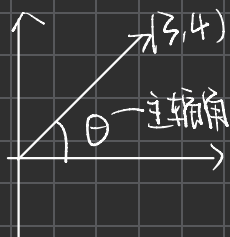
⑤ $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$, $|\frac{\bar{z}_1}{z_2}| = \frac{\bar{z}_1}{z_2}$

辐角与主辐角

1. 主辐角 θ

$$\theta \in (-\pi, \pi]$$

主: $\arg z = \frac{\theta}{\pi} \pi$

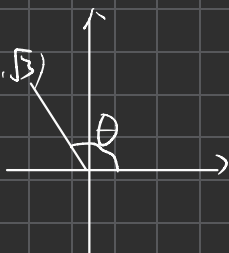


2. 辐角 = 主辐角 + $2k\pi$ ($k \in \mathbb{Z}$)

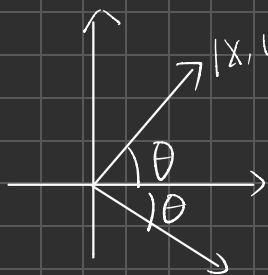
eg: $z = -1 + \sqrt{3}i$

辐角: $\theta = \frac{2\pi}{3} + 2k\pi$ (1, $\sqrt{3}$)

$\arg z = \frac{\theta}{\pi} \pi + 2k\pi$



题型 1: 计算辐角 | 主 $\arg z =$ 辐 $\arg z = \arg z + 2k\pi$



① 向量在一、四象限

$$\theta = \arctan \frac{y}{x}$$

③ 在三象限

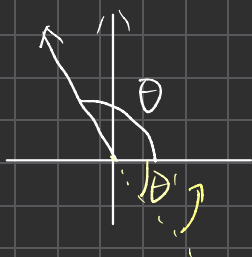


求取 θ' 而非所需 θ
 $\therefore \theta = \arctan \frac{y}{x} - \pi$

② 向量在二象限

用 $\theta = \arctan \frac{y}{x}$ 求出 θ' 而非所需 θ

$$\therefore \theta = \arctan \frac{y}{x} + \pi$$



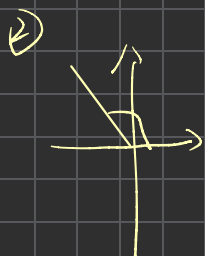
求 $\text{Arg}(2-2i)$ 及 $\text{Arg}(-3+4i)$

① $\text{Arg}(2-2i)$ $\arctan \frac{y}{x}$



$$\arg(2-2i) = \arctan \frac{-2}{2} = -\frac{\pi}{4}$$

$$\text{Arg}(2-2i) = -\frac{\pi}{4} + 2k\pi$$



$$\arg(-3+4i) = \arctan \frac{y}{x} + \pi$$

$$\text{Arg}(-3+4i) = \arctan \frac{y}{x} + (2k+1)\pi$$

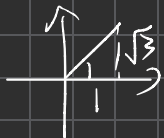
复数3种表达形式

代数: $x + yi$

三角: $r(\cos\theta + i\sin\theta)$ } r 模长

指数: $re^{i\theta}$ } θ 辐角

eg: $z = 1 + \sqrt{3}i$



$r = \sqrt{3+1} = 2$

$\theta = \frac{\pi}{3}$

$\therefore$ 三角 $= 2(\cos \frac{\pi}{3} + i\sin \frac{\pi}{3})$

$= 2e^{i\frac{\pi}{3}}$

$ze^{\frac{\pi}{3}i}$

$2(\cos \frac{\pi}{3} + i\sin \frac{\pi}{3})$

题型2: 代数换指数

方法1: 画图找 r 与 θ (如上)

方法2: 凑三角形式

$$: \frac{r \cos \theta}{x} + \frac{r \sin \theta}{y i} : \text{三角恒等作代换}$$

eg: 将复数 $1 - \cos \varphi + i \sin \varphi$ ($0 < \varphi \leq \pi$) 化为指数形式

$$\text{重要补充} \begin{cases} 1 - \cos \varphi = 2 \sin^2 \frac{\varphi}{2} \\ 1 + \cos \varphi = 2 \cos^2 \frac{\varphi}{2} \\ \sin \varphi = 2 \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} \end{cases} \quad (\text{半角公式})$$

$$\begin{aligned} \text{解: } & 2 \sin^2 \frac{\varphi}{2} + i (2 \cos \frac{\varphi}{2} \sin \frac{\varphi}{2}) \\ &= 2 \sin \frac{\varphi}{2} (\sin \frac{\varphi}{2} + i \cos \frac{\varphi}{2}) \\ &= \underbrace{2 \sin \frac{\varphi}{2}}_r \left[\cos \left(\frac{\pi}{2} - \frac{\varphi}{2} \right) + i \sin \left(\frac{\pi}{2} - \frac{\varphi}{2} \right) \right] \end{aligned}$$

$$\begin{aligned} \therefore z &= e^{i(\frac{\pi}{2} - \frac{\varphi}{2})} \cdot r \\ &= 2 \sin \frac{\varphi}{2} e^{i(\frac{\pi}{2} - \frac{\varphi}{2})} \end{aligned}$$

复数乘除 (模乘除, 角加减)

$$z_1 = 2e^{\frac{\pi}{3}i}$$

$$z_2 = 3e^{\frac{\pi}{6}i}$$

$$\begin{aligned} z_1 \cdot z_2 &= 2e^{\frac{\pi}{3}i} \cdot 3e^{\frac{\pi}{6}i} \\ &= (2 \times 3) e^{i(\frac{\pi}{3} + \frac{\pi}{6})} \end{aligned}$$

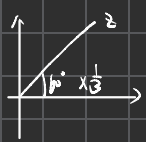
复数乘幂

$$z = 2e^{\frac{\pi}{3}i}$$

$$\begin{aligned} z^2 &= z \cdot z \\ &= 4e^{\frac{2\pi}{3}i} \end{aligned}$$

$$\begin{aligned} z^3 &= z \cdot z \cdot z \\ &= 8e^{\pi i} \end{aligned}$$

复数的n次根



$\sqrt[n]{z}$ $r = \sqrt[n]{r}$ ✓
 不知道其已经转了几圈
 ∴ 情况不唯一 ×

$\theta = 60^\circ \rightarrow \theta' = 20^\circ$
 $\theta = 60^\circ + 360^\circ \rightarrow \theta' = 140^\circ$
 $\theta = 60^\circ + 360^\circ + 360^\circ \rightarrow \theta' = 260^\circ$
 ∴ 10次根就有10种情况

$\therefore z = re^{i\theta} \quad \sqrt[n]{z} = \sqrt[n]{r} e^{i[\frac{\theta + 2k\pi}{n}]}$ $k = 0, 1, \dots, n-1$

eg: 计算 $\sqrt[3]{8}$

- 8

e^{ni} 8

☆ 看作 $z = -8 + 0i$
 $= -1 \times 8$
 $= 8e^{i\pi}$

$\sqrt[3]{8} e^{i(\frac{\pi + 2k\pi}{3})}$

$\therefore \sqrt[3]{8} = \sqrt[3]{8} \cdot e^{i(\frac{\pi + 2k\pi}{3})}$ $k = 0, 1, 2$

① $k=0 \therefore = 2 \cdot e^{i\frac{\pi}{3}} = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$
 $= \sqrt{3}i + 1$

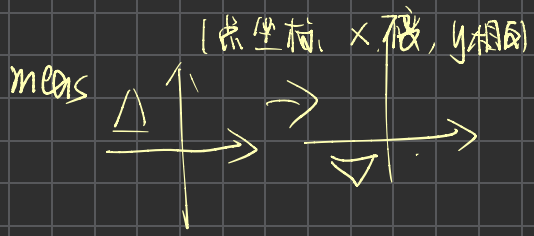
② $k=1 \therefore = 2 \cdot e^{i\frac{2\pi}{3}}$
 $= -2$

③ $k=2 \therefore = 2 \cdot e^{i\frac{4\pi}{3}}$
 $= 2(-\frac{\sqrt{3}}{2} + i \cdot -\frac{1}{2})$
 $= -\sqrt{3} - i$

复变函数——单值

$w = f(z)$

例: $f(z) = \bar{z}$, $z = x + yi$
 $\therefore f(z) = x - yi$ ↙ y轴取过来



题型：图像变化

例：设有函数 $w = z^2$ 试问它把 z 平面上的下列曲线分别变成 w 平面上的何种曲线

(1) 以原点为圆心, z 为半径, 第一象限里的圆弧

(2) 倾角 $\theta = \frac{\pi}{3}$ 的直线 (可以看成两条射线 $\arg z = \frac{\pi}{3}$ 及 $\arg z = \pi + \frac{\pi}{3}$)

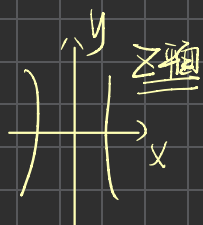
(3) 双曲线 $x^2 - y^2 = 4$

① 设 $z = x + yi$ (输入)

$w = u + vi$ (输出)

② z 图像表示

③ u, v 轴



$$f(z) = z^2$$

$$f(z) = (x + yi)^2 = x^2 - y^2 + 2xyi$$

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

$$\text{双曲线 } x^2 - y^2 = 4$$

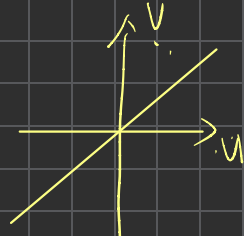
$$\therefore u = 4$$



变式：图像变为 $x^2 - y^2 = 2xy$

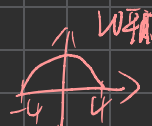
$$\therefore \begin{cases} u = x^2 - y^2 = 2xy \\ v = 2xy \end{cases}$$

$$\therefore u = v$$



做法2：① 设 $z = re^{i\theta}$
 $w = Re^{i\phi}$

1) 设 $z = re^{i\theta}$ $w = Re^{i\phi}$



② z 图像表示

③ $r, \theta \Leftrightarrow R, \phi$

$$w = z^2 = \frac{r^2}{R} e^{2i\theta} = e^{2i\phi} \therefore \begin{cases} R = r^2 = 4 \\ \phi = 2\theta = 0 < \phi < \frac{\pi}{3} \end{cases}$$

$$2) z = re^{i\theta} \quad w = Re^{i\varphi}$$

$$w|z|^2 \rightarrow r^2 e^{2i\theta} = Re^{i\varphi}$$

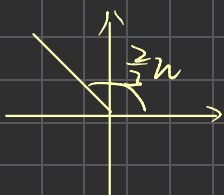
$$\begin{cases} R = r^2 \\ \varphi = 2\theta \end{cases} \quad \text{由题 } 0 \leq r < +\infty \text{ (射线)}$$

$$\theta = \frac{2\pi}{3} \text{ 或 } \theta = \pi + \frac{2\pi}{3}$$

$$\therefore 2\pi < +\infty$$

$$\varphi = \frac{2\pi}{3} \text{ 或 } 2\pi + \frac{2\pi}{3}$$

w平面



极限存在性

$$f(z) \quad z \rightarrow z_0 \quad f(z_0) = ?$$

$$\text{实部: } \lim_{\substack{(x,y) \rightarrow (x_0,y_0) \\ (x,y) \in E}} u(x,y) = a$$

$$\text{虚部: } \lim_{\substack{(x,y) \rightarrow (x_0,y_0) \\ (x,y) \in E}} v(x,y) = b$$

题型证明 极限不存在 (不同路径 趋近不同的值)

证明函数 $f(z) = \frac{z}{z+1}$ ($z \neq -1$) 当 $z \rightarrow 0$ 时 极限不存在

Step ① 设数 $z = x + yi$ ② 实、虚部 - 极限不存在

$$z \rightarrow 0 \quad (x,y) \rightarrow 0 \quad f = \frac{x+yi}{x-yi} = \frac{x^2-y^2+2xyi}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} + \frac{2xyi}{x^2+y^2}$$

实部不存在 $y=x: \frac{x^2-x^2}{x^2+x^2} = 0$
虚部 $y=2x: \frac{2x^2}{x^2+4x^2} = \frac{2}{5}$
 $\therefore$ 不样不存在

解析函数的柯西黎曼方程

一、可微与可导

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

题型：判断函数不可导(微)

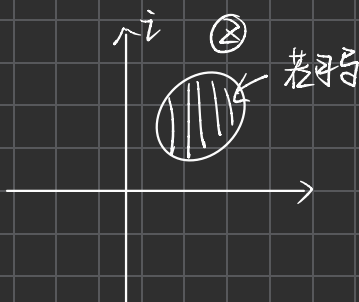
eg: 讨论函数 $f(z) = \bar{z}$ 在 z 平面上处处不可导

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z}$$

与之前一题一样
不存在 不可微

二、导数 法则与实数一致

三、解析性 (C-R 方程)



在有圆域区域可导就是解析

四、解析的充要条件：满足 C-R 方程

eg: $f(z) = u + v i$

$u(x, y) \quad v(x, y)$

$$C-R \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

题型：判断函数的解析性

eg: 讨论 $f(z) = |z|^2$ 的解析性

设 $z = x + y i \quad f(z) = u + v i$

$z f(z) = x^2 + y^2 \quad \therefore u = x^2 + y^2, v = 0$

$$C-R \begin{cases} \frac{\partial u}{\partial x} = 2x & \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} = 2y & \frac{\partial v}{\partial x} = 0 \end{cases} \quad \therefore \text{不解析}$$

eg: 讨论函数 $f(z) = x^2 - iy$ 的可微性与解析性

可导, 用 $\lim$

$$f(z) = x^2 - iy, \quad f(z+\Delta z) = (x+\Delta x)^2 - i(y+\Delta y) = x^2 + 2x\Delta x + \Delta x^2 - iy - i\Delta y$$

$$\lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - iy - i\Delta y - (x^2 - iy)}{\Delta x + i\Delta y} = \lim_{\Delta z \rightarrow 0} \frac{2x\Delta x + \Delta x^2 - i\Delta y}{\Delta x + i\Delta y}$$

$$z = x + iy$$

$$\begin{cases} u = x^2 \\ v = -y \end{cases} \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1$$

$$\Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}, \quad 0 = 0$$

只有一条线
解析
不解析

初等解析函数

一. 指数函数

$$e^z = e^{x+iy} = e^x (\cos y + i \sin y)$$

性质: ① $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$

② $(e^z)' = e^z$

③ $e^z \neq 0$

④ e^z 以 $2\pi i$ 为周期

二. 三角函数

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

性质: ① $(\cos z)' = -\sin z$

② $(\sin z)' = \cos z$

③ $\sin z$ 奇 $\cos z$ 偶

④ 以 2π 为周期

⑤ $|\sin z|$ 与 $|\cos z|$ 无界

补充: $\sin z = \sin(x+iy)$

① $y=0 \Rightarrow \sin x$

② $x=0 \Rightarrow \sin yi = \sin hy$

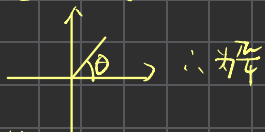
初等复函数

给1 赋值

1. 辐角函数 $\text{Arg} z = \arg z + 2k\pi$

例: $\text{Arg}(1+i) = \arg(1+i) + 2k\pi$

在坐标上对应 (1,1)



$$\arg = \frac{\pi}{4} + 2k\pi \quad k \in \mathbb{Z}$$

2. 对数函数 $\text{Ln} z = \ln z + 2k\pi i \quad k \in \mathbb{Z}$

辐

$$\ln z = \ln|z| + i \arg z$$

$$\text{Ln} z = \ln|z| + i \arg z + 2k\pi i$$

$\text{Ln} z$ 是 e^z 的反函数

周期 $2k\pi i$ (周)

例: $\text{Ln}(3+4i)$

$$\text{Ln}(3+4i) = \ln|3+4i| + 2k\pi i$$

$$= \ln 5 + i \arctan \frac{4}{3} + 2k\pi i$$

$$\arg z = \arctan \frac{4}{3}$$

3. 根式函数 (开 n 次根 取 n 个值)

$$\sqrt[3]{-8} = \begin{cases} -2 \\ 1-\sqrt{3}i \\ 1+\sqrt{3}i \end{cases}$$

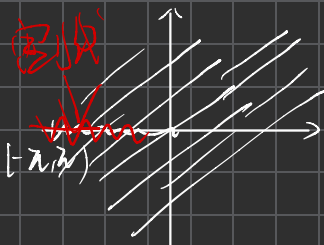
$$\sqrt[n]{z} = \sqrt[n]{|z|} e^{\frac{i \arg z + 2k\pi}{n}}$$

多值函数团图像

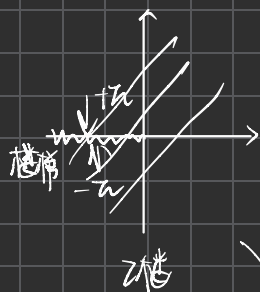
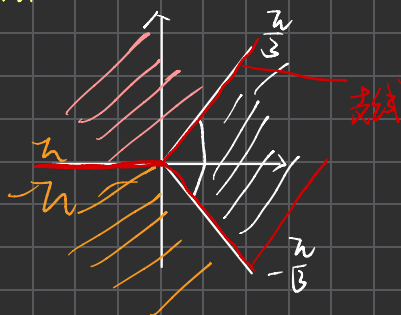
以 $f(z) = \sqrt[n]{z}$ 例

(模开 n 次方, 角取 i/n)

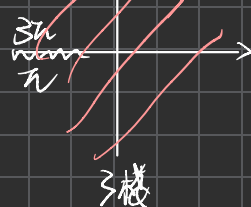
割破平面法



$\sqrt[n]{z}$ (角取 i/n)
模开 n 次方

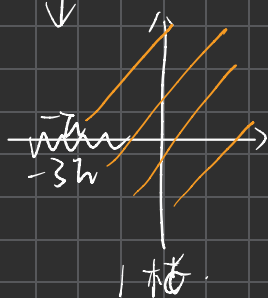


$M + 2\pi i$



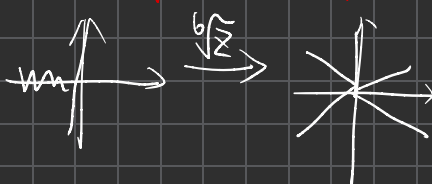
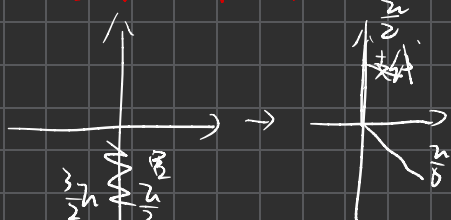
第 n 张纸
记为一张纸

$M - 2\pi i$



PS: 割线不同支线也不同

n 次根号有 n 个支线



奇点与支点 ☆ (不解析)

奇点：分母为0的点

eg: $f(z) = \frac{1}{z}$ $z=0$

$f(z) = \frac{1}{z^2}$ $z=0$

$f(z) = \frac{1}{z^2 - 3z - 4} = \frac{1}{(z-4)(z+1)} \quad \begin{cases} z=4 \\ z=-1 \end{cases}$

，变换后可连接两条直线

复变函数的积分

解法一：直接积

eg: 令 C 表示连接点 a 及 b 的任一曲线，试证：

1) $\int_C dz = b - a$ 2) $\int_C z dz = \frac{1}{2}(b^2 - a^2)$

取 C 为 $a \rightarrow b$ 直线 $\therefore \int_C = \int_a^b$

$= z \Big|_a^b$
 $= b - a$

$= \frac{z^2}{2} \Big|_a^b$
 $= \frac{b^2 - a^2}{2}$

解法二：令 x, y

eg: 计算积分 $\int_C (x-y+iz^2) dz$ ，积分路径 C 为由 $0 \rightarrow 1+iz$ 直线段

理论： $\int_C f(z) dz$

$f(z) = u + vi$

$z = x + yi$

两边微分

$dz = dx + i dy$

$\int_C f(z) dz = \int_C (u + vi)(dx + i dy)$
 $= \int_C u dx - v dy + i \left(\int_C v dx + u dy \right)$

解： $\int_C (x-y+iz^2)(dx + i dy)$

$= \int_C (x-y) dx - x^2 dy + i \int_C x^2 dx + (x-y) dy$

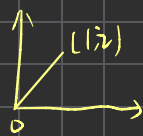
$+ i \int_C x^2 dx + (x-y) dy$

$= -\int_0^1 t^2 dt + i \int_0^1 t^2 dt$

$= -\frac{1}{3} + \frac{1}{3} i$

参数方程

$\begin{cases} x = t \\ y = t \end{cases}$
 $t \in (0, 1)$



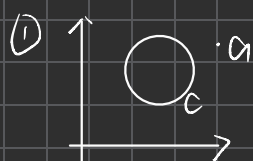
柯西积分

1. $\int_C f(z) dz$, C 是闭环

$f(z)$ 解析 则 $\int_C f(z) dz = 0$

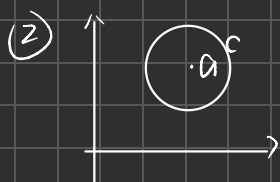
2. $\int_C \frac{1}{z-a} dz$

奇点 $z=a$ (不解析) 函数在奇点不解析



a 在 C 外
或在 C 上 $= 0$

条件 { 奇点是一级
除了 $\frac{1}{z-a}$ 没有 z
奇点在



a 在 C 内 $= 2\pi i$

eg: $\int_{|z|=1} \frac{1}{z-z} dz = 0$



$\int_{|z|=1} \frac{1}{z} dz$



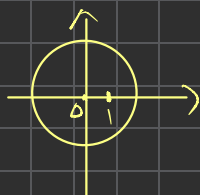
$= 2\pi i$

$\begin{aligned} \int_C &= \int_{\Gamma_1} + \int_{\Gamma_2} \\ &= \int_{\Gamma_1} \frac{1}{z-1} + \frac{1}{z} dz \\ &\quad + \int_{\Gamma_2} \frac{1}{z-1} + \frac{1}{z} dz \\ &= 0 + 2\pi i + 2\pi i \\ &= 4\pi i \end{aligned}$

(1 在 Γ_1 外 $= 0$)
(0 在 Γ_2 外 $= 0$)

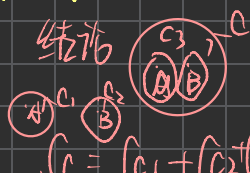
计算积分 $\int \frac{z-1}{z^2-z} dz$, Γ 为包含圆 $|z|=1$ 在内的任何正向简单闭曲线

means $z > 1$ 的任何闭曲线



$\int_{\Gamma} \frac{z-1}{z^2-z} dz = \int_{\Gamma} \frac{1}{z-1} - \frac{1}{z} dz$

奇点 $z=0, z=1$



$\int_C = \int_{C_1} + \int_{C_2}$

$$3. \int_C \frac{1}{z-a} dz \quad \begin{cases} 2\pi i & n=1 \\ 0 & n \neq 1 \end{cases} \quad \text{奇点为 } 1 \text{ 阶}$$

(假设 a 在 C 中)

$$4. \int_C \frac{f(z)}{z-a} dz = 2\pi i \cdot f(a)$$

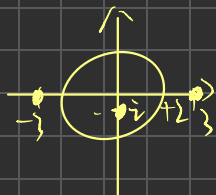
(假设 a 在 C 中)

$$\text{eg: } \int_{|z|=1} \frac{z^2+z}{z+\frac{1}{2}} dz = 2\pi i \cdot \left[\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) \right]$$



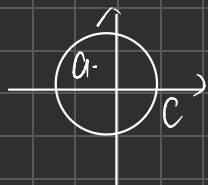
$$\text{例: 设 } C \text{ 为圆 } |s|=2 \quad \int_C \frac{\xi}{(9-\xi^2)(\xi+i)} d\xi =$$

$$\text{奇点: } \xi = \pm 3, \xi = -i$$

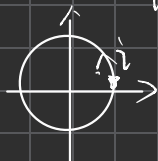


$$\begin{aligned} \int_C \frac{\xi}{(9-\xi^2)(\xi+i)} d\xi &= \int \frac{1}{\xi+i} \cdot \frac{\xi}{(9-\xi^2)} d\xi \\ &= 2\pi i \cdot \frac{-i}{9-1i^2} = \frac{\pi}{5} \end{aligned}$$

$$5. \int_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} \cdot f^{(n-1)}(a) \quad z=a$$



$$\text{例: } \int_C \frac{\cos z}{(z-i)^3} dz, \quad C \text{ 为绕 } i \text{ 线}$$



$$\begin{aligned} &= \frac{2\pi i}{(3-1)!} \cdot f'' \cos z \big|_{z=i} \\ &= -\pi i \cos z \big|_{z=i} = -\pi i \cos i \end{aligned}$$

定理①: 解析函数无论导几次都解析

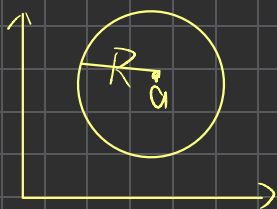
(2) $f(z) = U(x, y) + iV(x, y)$ 在 D 内解析的充要条件是

(1) U_x, U_y, V_x, V_y 在 D 内连续

(2) $U(x, y), V(x, y)$ 在 D 内满足 C-R 方程

(3) 柯西不等式: 设函数在 D 内解析, a 为 D 内一点, 以 a 为圆心作圆 $r: |z-a|=R$, 只要 r 及其内部 K 均含于 D , 则有

$$|f^{(n)}(a)| \leq \frac{n! M(R)}{R^n}$$



$M(R) = \max_{|z-a|=R} |f(z)|$ 在复平面上解析

★例: 设 $f(z)$ 是整函数, n 为正整数, 讨论: 当 $\lim_{z \rightarrow \infty} \frac{f(z)}{z^n} = 0$ 时, $f(z)$ 至多是 $n-1$ 次多项式

$$\Rightarrow f^{(n)}(z) = 0$$

$$|f^{(n)}(z)| \leq \frac{n! M(R)}{R^n} \quad \text{不妨设 } R > |z|$$

调和函数

$$\varphi(x,y), \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad (\varphi_{xx} + \varphi_{yy} = 0)$$

$\Rightarrow \varphi(x,y)$ 是调和函数

$f(z) = u + vi$ 解析

$\downarrow \quad \downarrow$

$$\frac{u(x,y)}{\Delta u = 0}$$

$$\frac{v(x,y)}{\Delta v = 0}$$

则 u 与 v 都是调和函数, 互为共轭调和函数.

1. 已知 u 求 v

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy + C$$

2. 已知 v 求 u

$$u(x,y) = \int_{(x_0,y_0)}^{(x,y)} v_y dx - v_x dy + C$$

eg: 验证 $u(x,y) = x^3 - 3xy^2$ 是 z 平面上的调和函数并求以 $u(x,y)$ 为实部的解析函数 $f(z)$, 使满足 $f(i) = i$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \text{为调和函数}$$

满足条件 (1) 已知 u 求 v

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy + C$$

$$= \int_{(0,0)}^{(x,y)} 6xy dx + (3x^2 - 3y^2) dy$$

折两折法)

$$\begin{aligned} &= \int_{(0,0)}^{(x,0)} \dots + \int_{(x,0)}^{(x,y)} \dots \\ &= 0 + \int_{x_0}^{xy} 6xy dx + (3x^2 - 3y^2) dy \\ &= \int_0^y (3x^2 - 3y^2) dy \\ &= 3x^2 y - y^3 + C \end{aligned}$$

$$f(z) = u + vi$$

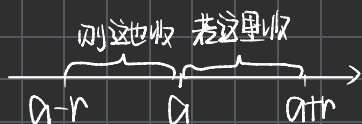
$$= x^3 - 3xy^2 + i(3x^2y - y^3 + C)$$

$$Rf(0) = i \quad \therefore C = 1$$

$$x=y=0$$

$$\therefore f(z) = x^3 - 3xy^2 + i(3x^2y - y^3 + 1)$$

幂级数的敛散性



求收敛半径

$$\sum_{n=0}^{\infty} C_n(z-a)^n = C_0 + C_1(z-a) + C_2(z-a)^2 + \dots$$

$$R = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{C_n}} = \lim_{n \rightarrow \infty} \left| \frac{C_n}{C_{n+1}} \right|$$

eg: $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

$$C_n = \frac{1}{n^2}$$

$$R = \lim_{n \rightarrow \infty} \sqrt[n]{n^2}$$

$$= \lim_{n \rightarrow \infty} n^{\frac{2}{n}}$$

$$= 1$$

$$\sum_{n=0}^{\infty} \cos(n) (z-1)^n$$

$$C_n = \cos n = \frac{e^{in} + e^{-in}}{2}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{\cos n}{\cos(n+1)} \right|$$

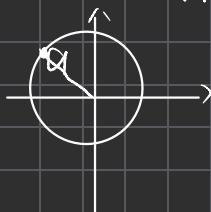
$$= \lim_{n \rightarrow \infty} \frac{e^n + e^{-n}}{e^{n+1} + e^{-(n+1)}} = \frac{1}{e}$$

泰勒定理: 设 $f(z)$ 是区域 D 内解析
 $a \in D$, 闭圆 $K: |z-a| \leq r$
 含于 D , 则 $f(z)$ 在 K 有唯一的展开
成幂级数

$$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$$

$$C_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{(\xi-a)^{n+1}} d\xi \quad (\text{积分})$$

$$= \frac{f^{(n)}(a)}{n!} \quad (\text{微分})$$



$$e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + \frac{(-1)^n z^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

$$= \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

eg: $\frac{1}{1+z} = \frac{1}{1-(-z)} = 1 - z + z^2 - z^3 + \dots$
 (当 $z \neq -1$) $= \sum_{n=0}^{\infty} (-z)^n$

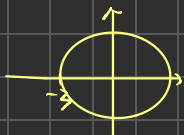
eg: 将 $\frac{e^z}{1-z}$ 在 $z=0$ 展开成幂级数 (幂级数在 $|z|<1$ 时收敛) $0<1$

$$\frac{e^z}{1-z} = e^z \cdot \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \cdot \sum_{n=0}^{\infty} z^n$$

eg: 试将函数 $f(z) = \frac{z}{z+2}$ 按 $z-1$ 的幂展开, 并指明其收敛范围 $z \neq -2$

$$f(z) = \frac{(z-1)+1}{(z-1)+3} \xrightarrow{\text{令 } z-1=u} = \frac{u+1}{u+3} = 1 - \frac{2}{u+3} = 1 - \frac{2}{3} \frac{1}{\frac{u}{3}+1}$$

换元



幂级数在 $|u|<3$ 收敛
 $z-1$

奇点 $u=-3$

$$= 1 - \frac{2}{3} \sum_{n=0}^{\infty} (-\frac{u}{3})^n$$

$$= 1 - \frac{2}{3} \sum_{n=0}^{\infty} (\frac{1}{3})^n (z-1)^n$$

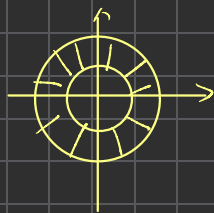
洛朗定理: 在圆环 $r < |z-a| < R$ ($r \geq 0, R \leq +\infty$) 内解析的函数 $f(z)$ 必可展成双级数

幂级数

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n$$

$$\text{其中 } C_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(s)}{(s-a)^{n+1}} ds \quad (n=0, \pm 1, \pm 2, \dots)$$

$|n|$ 可以为负, 允许有 $\frac{1}{z}$, 解析域为环
系不允许有 $\frac{1}{z}$



eg: $e^z + e^{\frac{1}{z}}$ 在 z 平面上只有奇点 $z=0$, 在其去心邻域 $0 < |z| < +\infty$ 内有洛朗展开式

$$e^z + e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(\frac{1}{z})^n}{n!} \rightarrow \frac{1}{z} + \frac{1}{z^2} + \dots$$



例 5.1 函数

$$f(z) = \frac{1}{(z-1)(z-2)}$$

在 z 平面上只有两个奇点: $z=1$ 及 $z=2$. 因此 z 平面被分成如下三个不相交的 $f(z)$ 的解析区域: (1) 圆 $|z|<1$, (2) 圆环 $1<|z|<2$, (3) 圆环 $2<|z|<+\infty$. 试分别在此三个区域内求 $f(z)$ 的展式.

(1) $|z|<1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \sum_{n=0}^{\infty} z^n - \frac{1}{2} \sum_{n=0}^{\infty} (\frac{z}{2})^n$$

(比0还大为平凡, 11不能0)

(2)

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= -\frac{1}{1-z} + \frac{1}{2} \sum_{n=0}^{\infty} (\frac{z}{2})^n$$

$$= -\frac{1}{1-z} + \frac{1}{2} \sum_{n=0}^{\infty} (\frac{1}{2})^n (z-1)^n$$

根 $\frac{1}{z}$

(3)

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z} \left(\frac{1}{1-\frac{1}{z}} \right) - \frac{1}{z} \left(\frac{1}{1-\frac{2}{z}} \right)$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} (\frac{1}{z})^n - \frac{1}{z} \sum_{n=0}^{\infty} (\frac{2}{z})^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^{n+1}}{z^{n+1}}$$

都在该处



用洛朗法
求出洛朗

圆2卡key

孤立奇点的分类

1. 可去奇点 (分母可约掉)

定理 5.3 如果 a 为函数 $f(z)$ 的孤立奇点, 则下列三条是等价的, 因此, 它们中的任何一条都是可去奇点的特征.

- (1) $f(z)$ 在点 a 的主要部分为零.
- (2) $\lim_{z \rightarrow a} f(z) = b (\neq \infty)$.
- (3) $f(z)$ 在点 a 的某去心邻域内有界.

eg: 说明 $z=0$ 为 $\frac{e^z-1}{z}$ 的可去奇点

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \dots$$
$$\text{代入: } \frac{1 + z + \frac{z^2}{2!} + \dots}{z} = \frac{1}{z} + 1 + \frac{z}{2} + \dots$$

2. 极点 (分母约不掉)

定理 5.5 函数 $f(z)$ 的孤立奇点 a 为极点的充要条件是

$$\lim_{z \rightarrow a} f(z) = \infty.$$

eg: 函数 $f(z) = \frac{\sqrt{z+1}}{(z-1)(z+1)^2}$ 以 $z=1$ 为一阶极点, $z=-\frac{1}{2}$ 为二阶极点

3. 本质奇点

3. 本质奇点

5. 本质奇点

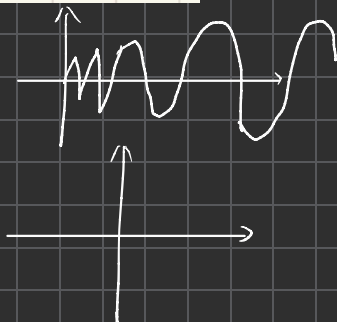
定理 5.6 函数 $f(z)$ 的孤立奇点 a 为本质奇点的充要条件是

$$\lim_{z \rightarrow a} f(z) \neq \begin{cases} b (\text{有限数}), \\ \infty, \end{cases} \text{ 即 } \lim_{z \rightarrow a} f(z) \text{ 不存在.}$$

这可由定理 5.3(2) 及定理 5.5 得到证明.

eg: $\sin \frac{1}{z}$
 $\frac{1}{z} \rightarrow 0$

eg: $e^{\frac{1}{z}}$ 周期



7.2 偏导数与全微分

7.2.1 偏导数的概念

1 偏导数的定义

(1) $f(x,y)$ 在点 $P_0(x_0,y_0)$ 处的偏导数

定义 7.2.1 设函数 $z = f(x,y)$ 在点 (x_0,y_0) 的某一邻域内有定义，当 y 固定在 y_0 而 x 在 x_0 处有增量 Δx 时，相应地函数有增量

如果
$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \quad (1) \text{ 存在,}$$

则称此极限为函数 $z = f(x, y)$ 在点 (x_0, y_0) 处对 x 的偏导数, 记为

$$\left. \frac{\partial z}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0}}, \quad \left. \frac{\partial f}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0}}, \quad z'_x \bigg|_{\substack{x=x_0 \\ y=y_0}} \text{ 或 } f'_x(x_0, y_0).$$

例如, 极限(1)可以表示为

$$f'_x(x_0, y_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

同理可定义函数 $z = f(x, y)$ 在点 (x_0, y_0) 处对 y 的偏导数, 为

$$\lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

记为 $\left. \frac{\partial z}{\partial y} \right|_{\substack{x=x_0 \\ y=y_0}}$, $\left. \frac{\partial f}{\partial y} \right|_{\substack{x=x_0 \\ y=y_0}}$, $\left. z_y' \right|_{\substack{x=x_0 \\ y=y_0}}$ 或 $f_y'(x_0, y_0)$.

$$\text{即 } f_y'(x_0, y_0) = \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

(2) 偏导函数

如果函数 $z = f(x, y)$ 在区域 D 内任一点 (x, y) 处对 x 的偏导数都存在, 那么这个偏导数就是 x 、 y 的函数, 它就称为函数 $z = f(x, y)$ 对自变量 x 的偏导数,

记作 $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$, z'_x 或 $f'_x(x, y)$.

同理可以定义函数 $z = f(x, y)$ 对自变量 y

的偏导数, 记作 $\frac{\partial z}{\partial y}$, $\frac{\partial f}{\partial y}$, z'_y 或 $f'_y(x, y)$.

注: $f'_x(x_0, y_0) = f'_x(x, y) \Big|_{\substack{x=x_0 \\ y=y_0}}$

$f'_y(x_0, y_0) = f'_y(x, y) \Big|_{\substack{x=x_0 \\ y=y_0}}$

(3) 偏导数概念可推广到二元以上的函数

如 $u = f(x, y, z)$ 在 (x, y, z) 处

$$f'_x(x, y, z) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x},$$

$$f'_y(x, y, z) = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y},$$

$$f'_z(x, y, z) = \lim_{\Delta z \rightarrow 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}.$$

2. 偏导数的计算

仍然是一元函数的求导公式和求导法则，对某一个自变量求偏导时，其余的自变量看作常量。

例 1 求 $z = x^2 + 3xy + y^2$ 在点(1,2)处的偏导数.

解

$$\frac{\partial z}{\partial x} = 2x + 3y \quad \frac{\partial z}{\partial y} = 3x + 2y$$

$$\frac{\partial z}{\partial x} \Big|_{(1,2)} = 8 \quad = 7$$

例2. $u = x^{y^z}$, 求 u'_x, u'_y, u'_z

解: $u'_x = y^z \cdot x^{y^z-1}$

对幂指数函数两边取对数

$$\ln u = y^z \ln x \quad \frac{u'_y}{u} = zy^{z-1} \ln x$$

$$\Rightarrow u'_y = \frac{z}{y} \ln x \cdot y^z \cdot x^{y^z}$$

$$\text{或: } u'_z = \frac{\partial}{\partial z} (e^{y^z \ln x}) = x^{y^z} \cdot y^z \cdot \ln y \ln x$$

例4

$$f(x, y) = x^2 + (y - 1) \arcsin \sqrt{\frac{x}{y}},$$

求 $f'_x(x, 1)$, $f'_x(2, 1)$.

解

$$f(x, 1) = x^2, \quad f'_x(x, 1) = \frac{df(x, 1)}{dx} = 2x;$$

$$f'_x(2, 1) = 4$$

注：(2) 求 $f'_x(x_0, y_0)$ 时，可先将 y_0 代入得

$f(x, y_0) = \varphi(x)$ ，再求 $\frac{d\varphi}{dx}$ ，即 $\frac{d\varphi}{dx} = \frac{df(x, y_0)}{dx}$ ，

最后再将 x_0 代入.

例 5 设 $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

求 $f(x, y)$ 的偏导数. 分别求 f'_x 与 f'_y

解 当 $(x, y) \neq (0, 0)$ 时,

$$f'_x(x, y) = \frac{y(x^2 + y^2) - 2x \cdot xy}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2},$$

$$f'_y(x, y) = \frac{x(x^2 + y^2) - 2y \cdot xy}{(x^2 + y^2)^2} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2},$$

注:(3)求分界点、不连续点处的偏导数要用定义求

当 $(x, y) = (0, 0)$ 时, 按定义可知:

$$f_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0,$$

$$f_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y} = 0,$$

$$f_x(x, y) = \begin{cases} \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases},$$

$$f_y(x, y) = \begin{cases} \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

3 偏导数存在与连续的关系

一元函数中在某点可导 $\longrightarrow$ 连续,

多元函数中在某点偏导数存在 $\xrightarrow{?}$ 连续,

例如,函数
$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases},$$

依定义知在 $(0,0)$ 处, $f'_x(0,0) = f'_y(0,0) = 0$.

但函数在该点处并不连续. 偏导数存在 $\nrightarrow$ 连续.

7.2.2 偏导数的几何意义

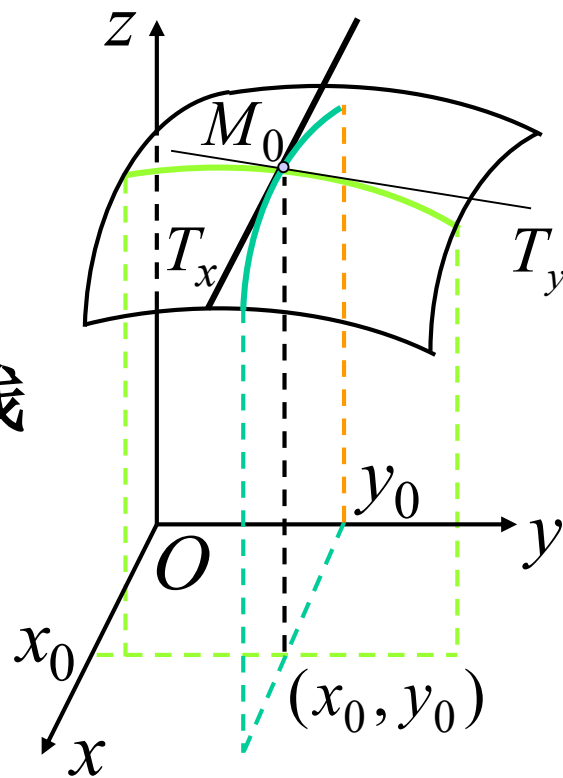
$$\left. \frac{\partial f}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0}} = \left. \frac{d}{dx} f(x, y_0) \right|_{x=x_0}$$

是曲线 $\begin{cases} z = f(x, y) \\ y = y_0 \end{cases}$ 在点 M_0 处的切线

$M_0 T_x$ 对 x 轴的斜率.

$$\left. \frac{\partial f}{\partial y} \right|_{\substack{x=x_0 \\ y=y_0}} = \left. \frac{d}{dy} f(x_0, y) \right|_{y=y_0}$$

是曲线 $\begin{cases} z = f(x, y) \\ x = x_0 \end{cases}$ 在点 M_0 处的切线 $M_0 T_y$ 对 y 轴的斜率.



7.2.3 高阶偏导数

函数 $z = f(x, y)$ 的二阶偏导数为

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}''(x, y),$$

纯偏导

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}''(x, y)$$

$$\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}''(x, y),$$

混合偏导

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}''(x, y)$$

类似可以定义更高阶的偏导数.

例如, $z = f(x, y)$ 的三阶偏导数为

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 z}{\partial x^2} \right) = \frac{\partial^3 z}{\partial x^3} = f'''_{xxx} = f'''_{x^3}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial^2 z}{\partial x^2} \right) = \frac{\partial^3 z}{\partial x^2 \partial y} = f'''_{xyx} = f'''_{x^2 y}$$

二阶及二阶以上的偏导数统称为高阶偏导数.

问题： 混合偏导数都相等吗？

具备怎样的条件才能使混合偏导数相等？

定理 7.2.1 如果函数 $z = f(x, y)$ 的两个二阶混合偏导数 $\frac{\partial^2 z}{\partial y \partial x}$ 及 $\frac{\partial^2 z}{\partial x \partial y}$ 在区域 D 内连续，那么在该区域内这两个二阶混合偏导数必相等.

例 7 验证函数 $u(x, y) = \ln \sqrt{x^2 + y^2}$ 满足拉普拉斯方程 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

解 $\because \ln \sqrt{x^2 + y^2} = \frac{1}{2} \ln(x^2 + y^2),$

$$\therefore \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2},$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0.$$

证毕.

7.2.4 全微分

1. 增量、全增量

由一元函数微分学中增量与微分的关系得

$$f(x + \Delta x, y) - f(x, y)$$

$$f(x, y + \Delta y) - f(x, y)$$

$$\approx f'_x(x, y)\Delta x$$

$$\approx f'_y(x, y)\Delta y$$

二元函数

对 x 和对 y 的偏增量

二元函数

对 x 和对 y 的偏微分

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \quad (1)$$

叫做函数在点 (x, y) 对应于自变量增量 Δx 、 Δy 的全增量。

2. 全微分的定义

定义 7.2.2 如果函数 $z = f(x, y)$ 在点 (x, y) 的全增量 $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ 可以表示为 $\Delta z = A\Delta x + B\Delta y + o(\rho)$, 其中 A, B 不依赖于 $\Delta x, \Delta y$ 而仅与 x, y 有关,

$\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, 则称函数 $z = f(x, y)$ 在点 (x, y) 可微分, $A\Delta x + B\Delta y$ 称为函数 $z = f(x, y)$ 在点 (x, y) 的**全微分**, 记为 dz , 即 $dz = A\Delta x + B\Delta y$.

函数若在某区域 D 内各点处处可微分, 则称这函数在 D 内可微分.

如果函数 $z = f(x, y)$ 在点 (x, y) 可微分, 则函数在该点连续. 可微不一定连续

事实上 $\Delta z = A\Delta x + B\Delta y + o(\rho), \lim_{\rho \rightarrow 0} \Delta z = 0$

3 可微的必要条件: 可微偏导数一定存在

定理 6.2.2 (必要条件) 如果函数 $z = f(x, y)$ 在点 (x, y) 可微分, 则该函数在点 (x, y) 的偏导数 $\frac{\partial z}{\partial x}$ 、 $\frac{\partial z}{\partial y}$ 必存在, 且函数

$z = f(x, y)$ 在点 (x, y) 的全微分

$$dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y.$$

4. 偏导存在不是函数可微的充分条件

一元函数可微等价于可导。

而多元函数偏导存在不能推出可微。

$$\text{例如 } f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0 \end{cases}$$

$f(x, y)$ 在点 $P_0(0,0)$ 处偏导存在, 但 $f(x, y)$ 在点 P_0 处不连续。所以 $f(x, y)$ 在点 P_0 处一定不可微。

5. 函数可微的充分条件

定理 7.2.3 (充分条件) ③ 如果函数

$z = f(x, y)$ 的偏导数 $\frac{\partial z}{\partial x}$ 、 $\frac{\partial z}{\partial y}$ 在点 (x, y) 连续, 则

该函数在点 (x, y) 可微分.

① 可微 $\rightarrow$ 偏导数存在

② 偏导数存在 $\nrightarrow$ 可微 (不连续)

习惯上，记全微分为 $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$

全微分的定义可推广到三元及三元以上函数

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz.$$

6. 全微分的计算

方法:

(1) 先求 $f'_x(x,y)$ 、 $f'_y(x,y)$, 判断 $f(x,y)$ 的可微性
(利用充分条件)

$$(2) \quad dz = f'_x(x, y)dx + f'_y(x, y)dy$$

几类微分: (i) $P(x,y)$ 处的微分;

(ii) $P_0(x_0, y_0)$ 处的微分;

(iii) $P_0(x_0, y_0)$ 处且 dx, dy 给定时的微分

- 例1. (1) 计算 $z = x^2y + y^3$ 的全微分;
(2) 计算 $z = x^2y + y^3$ 在点(2, 1)处的全微分;
(3) 计算 $z = x^2y + y^3$ 在点(2, 1)处相应于
 $\Delta x = 0.1$, $\Delta y = -0.1$ 时的全微分。

解 (1) $\frac{\partial z}{\partial x} = 2xy$ $\frac{\partial z}{\partial y} = x^2 + 3y^2$

$$dz = 2xy dx + (x^2 + 3y^2) dy$$

(2) $\frac{\partial z}{\partial x} = 2xy$ $\frac{\partial z}{\partial y} = x^2 + 3y^2$

$$\text{代入}(2, 1) = 4 \qquad = 7$$

$$dz = 4dx + 7dy$$

(3) $\frac{\partial z}{\partial x} = 2xy$ $\frac{\partial z}{\partial y} = x^2 + 3y^2$

$$\text{代入}(2, 1) = 4 \qquad = 7$$

$$dz = 4 \times 0.1 + 7 \times -0.1 = -0.3$$

例 2 计算函数 $u = x + \sin \frac{y}{2} + e^{yz}$ 的全微分.

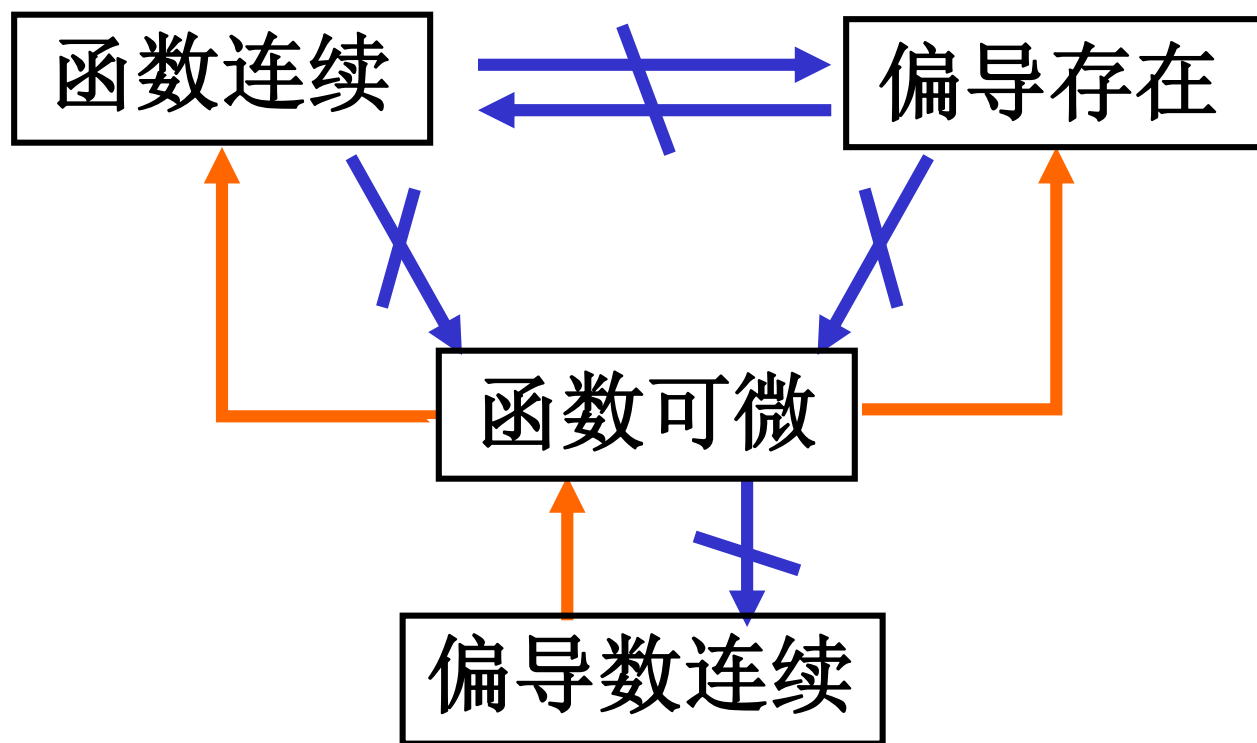
解: $\frac{\partial u}{\partial x} = 1$

$$\frac{\partial u}{\partial y} = \cos \frac{y}{2} \cdot \frac{1}{2} + e^{yz} \cdot z$$

$$\frac{\partial u}{\partial z} = e^{yz} \cdot y$$

$$du = dx + \left(\frac{1}{2} \cos \frac{y}{2} + ze^{yz} \right) dy + e^{yz} \cdot y dz$$

7 多元函数连续、可导、可微的关系



7.3 多元复合函数求导法

7.3.1 多元与一元的复合

1 自变量只有一个的情况

定理 7.3.1 如果函数 $u = \varphi(t)$ 及 $v = \phi(t)$ 都在点 t 可导, 函数 $z = f(u, v)$ 在对应点 (u, v) 具有连续偏导数, 则复合函数 $z = f[\varphi(t), \phi(t)]$ 在对应点 t 可导, 且其导数可用下列公式计算:

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt}.$$

证明 设 t 获得增量 Δt ,

则 $\Delta u = \varphi(t + \Delta t) - \varphi(t)$, $\Delta v = \phi(t + \Delta t) - \phi(t)$;

由于函数 $z = f(u, v)$ 在点 (u, v) 有连续偏导数

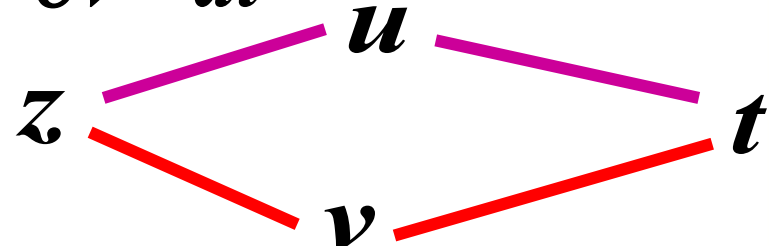
$$\Delta z = \frac{\partial z}{\partial u} \Delta u + \frac{\partial z}{\partial v} \Delta v + \varepsilon_1 \Delta u + \varepsilon_2 \Delta v,$$

当 $\Delta u \rightarrow 0$, $\Delta v \rightarrow 0$ 时, $\varepsilon_1 \rightarrow 0$, $\varepsilon_2 \rightarrow 0$

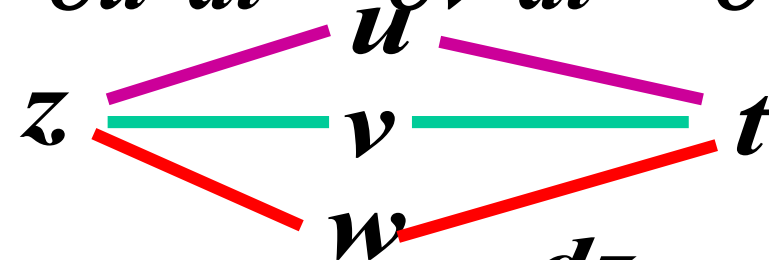
$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial u} \cdot \frac{\Delta u}{\Delta t} + \frac{\partial z}{\partial v} \cdot \frac{\Delta v}{\Delta t} + \varepsilon_1 \frac{\Delta u}{\Delta t} + \varepsilon_2 \frac{\Delta v}{\Delta t}$$

当 $\Delta t \rightarrow 0$ 时, $\Delta u \rightarrow 0$, $\Delta v \rightarrow 0$

$$\frac{\Delta u}{\Delta t} \rightarrow \frac{du}{dt}, \quad \frac{\Delta v}{\Delta t} \rightarrow \frac{dv}{dt},$$

$$\begin{aligned}\frac{dz}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} \\ &= f'_u \cdot \varphi'(t) + f'_v \cdot \phi'(t)\end{aligned}$$


注1 上定理的结论可推广到中间变量多于两个的情况. 例如 $z = f(u, v, w)$,

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} + \frac{\partial z}{\partial w} \frac{dw}{dt}$$


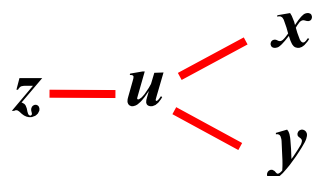
2: 以上公式中的导数 $\frac{dz}{dt}$ 称为**全导数**.

2 当仅有一个中间变量时

$z = f(u), u = \varphi(x, y)$ 则复合函数 $z = f[\varphi(x, y)]$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{df}{du} \cdot \frac{\partial \varphi}{\partial x} = f'(u) \cdot \varphi'_x,$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{df}{du} \cdot \frac{\partial \varphi}{\partial y} = f'(u) \cdot \varphi'_y$$



例 1 设 $z = uv + \sin t$ ，而 $u = e^t$ ， $v = \cos t$ ，

求全导数 $\frac{dz}{dt}$ 。

解

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial z}{\partial t} \\ &= ve^t - u \sin t + \cos t \\ &= e^t \cos t - e^t \sin t + \cos t \\ &= e^t (\cos t - \sin t) + \cos t.\end{aligned}$$

这种多元初等函数的偏导计算可不必用以上方法，而直接利用一元函数的求导法则计算。

例2 设 $z = \frac{y}{f(x^2 - y^2)}$, 其中 $f(u)$ 可导, 验证

$$\frac{1}{x} \frac{\partial z}{\partial x} + \frac{1}{y} \frac{\partial z}{\partial y} = \frac{z}{y^2}$$

证明

$$\frac{\partial z}{\partial x} = -\frac{yf' \cdot 2x}{f^2} \quad \frac{\partial z}{\partial y} = \frac{f - yf' \cdot (-2y)}{f^2}$$

$$\begin{aligned} & \frac{1}{x} \frac{\partial z}{\partial x} + \frac{1}{y} \frac{\partial z}{\partial y} \\ &= \frac{1}{x} \left(-\frac{yf' \cdot 2x}{f^2} \right) + \frac{1}{y} \left(\frac{f - yf' \cdot (-2y)}{f^2} \right) = \frac{z}{y^2} \end{aligned}$$

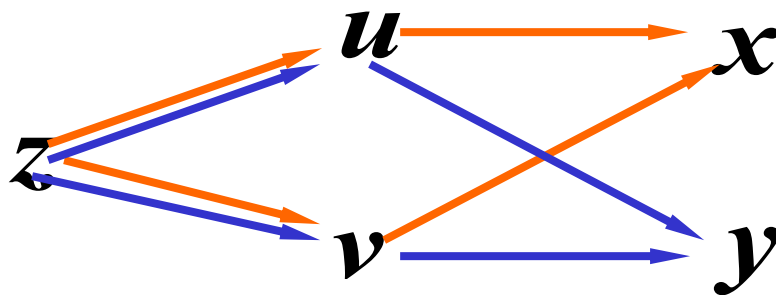
7.3.2 多元与多元的复合

定理6.3.1可推广到中间变量不是一元函数而是多元函数的情况： $z = f[\varphi(x, y), \phi(x, y)]$.

定理 7.3.2 如果 $u = \varphi(x, y)$ 及 $v = \phi(x, y)$ 都在点 (x, y) 具有对 x 和 y 的偏导数，且函数 $z = f(u, v)$ 在对应点 (u, v) 具有连续偏导数，则复合函数 $z = f[\varphi(x, y), \phi(x, y)]$ 在对应点 (x, y) 的两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.$$

链式法则如图示

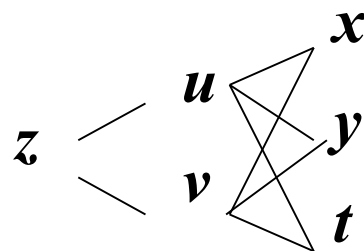
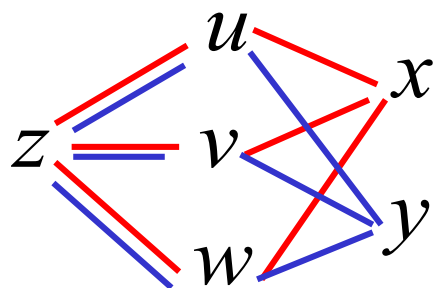


$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = f'_u \cdot \phi'_x + f'_v \cdot \phi'_x$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = f'_u \cdot \phi'_y + f'_v \cdot \phi'_y$$

链式法则：“连线相乘，分线相加”

注 1 上述公式可推广：中间变量及自变量的个数可增加或减少。例如



2 当复合函数中自变量与中间变量共存时

设 $z=f(u, x, y)$ 具有连续偏导数,

$u=\varphi(x, y)$ 具有导数,

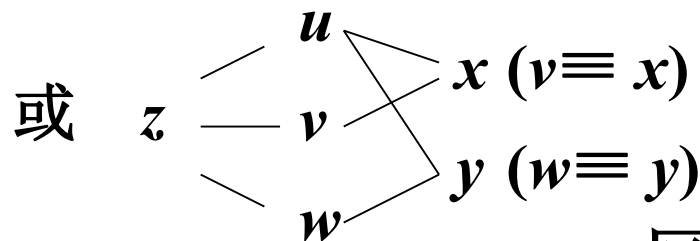
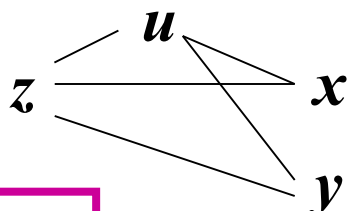
则复合函数 $z=f[\varphi(x, y), x, y]$

对 x, y 的偏导数为:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial x} = f'_u u'_x + f'_x$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial y} = f'_u u'_y + f'_y$$

变量关系图:



$$\boxed{\frac{\partial z}{\partial x}} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \boxed{\frac{\partial f}{\partial x}},$$

$$\boxed{\frac{\partial z}{\partial y}} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \boxed{\frac{\partial f}{\partial y}}.$$

区别类似

两者的区别

把复合函数 $z = f[\phi(x, y), x, y]$ 中的 y 看作不变而对 x 的偏导数

把 $z = f(u, x, y)$ 中的 u 及 y 看作不变而对 x 的偏导数

例 3 设 $z = e^u \sin v$, 而 $u = xy$, $v = x + y$,

求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.

解
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1 = e^u (y \sin v + \cos v)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= e^u \sin v \cdot x + e^u \cos v \cdot 1 = e^u (x \sin v + \cos v).$$

注 也可由 $z = e^{xy} \sin(x+y)$ 而直接对 x 、 y 求偏导。
注意两种方法的区别。

例4 $u = f(x, y, z) = e^{x^2 + y^2 + z^2}$

而 $z = x^2 \sin y$ 。求 $\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y}$

解: $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}$

$$= 2xe^{x^2 + y^2 + z^2} + 2ze^{x^2 + y^2 + z^2} \cdot 2x \sin y$$

$$= 2x(1 + 2x^2 \sin^2 y)e^{x^2 + y^2 + x^4 \sin^2 y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$= 2ye^{x^2 + y^2 + z^2} + 2ze^{x^2 + y^2 + z^2} \cdot x^2 \cos y$$

$$= 2(y + x^4 \sin y \cos y)e^{x^2 + y^2 + x^4 \sin^2 y}$$

7.3.3 多元复合函数的高阶偏导数

通过例题介绍多元复合函数的高阶偏导数

例 5 设 $w = f(x + y + z, xyz)$, f 具有二阶

连续偏导数, 求 $\frac{\partial w}{\partial x}$ 和 $\frac{\partial^2 w}{\partial x \partial z}$.

解 令 $u = x + y + z$, $v = xyz$;

记 $f'_1 = f_1 = \frac{\partial f(u, v)}{\partial u}$, $f''_{12} = f_{12} = \frac{\partial^2 f(u, v)}{\partial u \partial v}$,

同理有 $f'_2, f''_{11}, f''_{22}, f_2, f_{11}, f_{21}, f_{22}$.

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = f'_1 + yzf'_2; \\ & \quad (= f_1 + yzf_2);\end{aligned}$$

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z} (f_1' + yzf_2') = \frac{\partial f_1'}{\partial z} + yf_2' + yz \frac{\partial f_2'}{\partial z};$$

$$\frac{\partial f_1'}{\partial z} = \frac{\partial f_1'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_1'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{11}'' + xyf_{12}'';$$

$$\frac{\partial f_2'}{\partial z} = \frac{\partial f_2'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_2'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{21}'' + xyf_{22}'';$$

于是

$$\begin{aligned} \frac{\partial^2 w}{\partial x \partial z} &= f_{11}'' + xyf_{12}'' + yf_2' + yz(f_{21}'' + xyf_{22}'') \\ &= f_{11}'' + y(x+z)f_{12}'' + xy^2zf_{22}'' + yf_2'. \end{aligned}$$

例6 设 $z=g(2x+y)+f(2x-y, y\sin x)$, 求 z_{xy} 。

$$\text{解: } z_x = \frac{dg}{dw} \cdot \frac{dw}{dx} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= z g' + z f_1 + y \cos x f_2$$

$$z_{xy} = z g'' + z (f_{11}(-1) + f_{12} \sin x) + f_2 \cos x + y \cos x (f_{21}(-1) + f_{22} \sin x)$$

$$= z g'' (2x+y) - f_{11} + z f_{12} \sin x + f_2 \cos x - y f_{21} \cos x + y f_{22} \sin x \cos x$$

7.3.4 微分求导法 ——一阶微分形式不变性

设 $z=f(u,v)$ 具有连续的偏导数, 则有

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$$

又设 $u = \varphi(x, y), v = \phi(x, y)$ 也具有连续偏导时,

则复合函数 $z = f[\varphi(x, y), \phi(x, y)]$ 的全微分为:

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \\ &= \left(\frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \right) dx + \left(\frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \right) dy \end{aligned}$$

$$\begin{aligned}
 &= \frac{\partial z}{\partial u} \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + \frac{\partial z}{\partial v} \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) \\
 &= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv.
 \end{aligned}$$

由此可见，不论 z 是自变量 u 、 v 函数，或是中间变量 u 、 v 的函数，它的全微分形式是一样的，都是

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$$

这个性质叫全微分形式的不变性。

利用这一性质，可求复合函数、隐函数的偏导数。

7.4.1、 一个方程的情形

1. $F(x, y) = 0$

定理1. 设函数 $F(x, y)$ 在点 $P(x_0, y_0)$ 的某一邻域内满足

① 具有连续的偏导数;

② $F(x_0, y_0) = 0$;

③ $F_y(x_0, y_0) \neq 0$

则方程 $F(x, y) = 0$ 在点 x_0 的某邻域内可唯一确定一个连续函数 $y = f(x)$, 满足条件 $y_0 = f(x_0)$, 并有连续导数

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \quad (\text{隐函数求导公式})$$

定理证明从略, 仅就求导公式推导如下:

设 $y=y(x)$ 为方程 $F(x,y)=0$ 所确定的隐函数, 则

$$F(x, y(x)) \equiv 0$$

两边对 x 求导:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0, \quad (F'_x + F'_y \frac{dy}{dx} = 0)$$

由于 F_y 连续, 且 $F_y(x_0, y_0) \neq 0$, 所以存在 (x_0, y_0) 的一个邻域, 在这邻域内 $F_y \neq 0$, 于是得

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y}$$

在 $J \neq 0$ 的条件下,

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = -\frac{xu + yv}{x^2 + y^2}, \quad \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = \frac{yu - xv}{x^2 + y^2},$$

将所给方程的两边对 y 求导, 用同样方法得

$$\frac{\partial u}{\partial y} = \frac{xv - yu}{x^2 + y^2}, \quad \frac{\partial v}{\partial y} = -\frac{xu + yv}{x^2 + y^2}.$$

定理3. 设函数 $F(x, y, u, v), G(x, y, u, v)$ 满足:

① 在点 $P(x_0, y_0, u_0, v_0)$ 的某一邻域内具有连续偏导数;

② $F(x_0, y_0, u_0, v_0) = 0, G(x_0, y_0, u_0, v_0) = 0$;

③ $J|_P = \frac{\partial (F, G)}{\partial (u, v)} \Big|_P \neq 0,$

则方程组 $F(x, y, u, v) = 0, G(x, y, u, v) = 0$ 在点 (x_0, y_0) 的某一邻域内可唯一确定一组满足条件 $u_0 = u(x_0, y_0), v_0 = v(x_0, y_0)$ 的连续函数 $u = u(x, y), v = v(x, y)$, 且有偏导数公式:

若 $F(x, y)$ 的二阶偏导数也都连续,

则还可求隐函数的二阶导数

$$\begin{aligned}\frac{d^2 y}{dx^2} &= - \frac{F'_y(F''_{xx} + F''_{xy} \frac{dy}{dx}) - F'_x(F''_{yx} + F''_{yy} \frac{dy}{dx})}{F'^2_y} \\ &= - \frac{F''_{xx} F'^2_y - 2F''_{xy} F'_x F'_y + F'^2_x F''_{yy}}{F'^3_y}\end{aligned}$$

例1. 验证方程 $\sin y + e^x - xy - 1 = 0$ 在点 $(0,0)$ 某邻域可确定可导隐函数 $y = f(x)$, 并求

$$\left. \frac{dy}{dx} \right|_{x=0}, \quad \left. \frac{d^2 y}{dx^2} \right|_{x=0}$$

解: 令 $F(x, y) = \sin y + e^x - xy - 1$, 则

$$\textcircled{1} \quad F_x = e^x - y, \quad F_y = \cos y - x \text{ 连续};$$

$$\textcircled{2} \quad F(0,0) = 0;$$

$$\textcircled{3} \quad F_y(0,0) = 1 \neq 0,$$

由定理1可知, 在 $x = 0$ 的某邻域内方程存在可导的隐函数 $y = f(x)$, 且

$$F(x, y) = \sin y + e^x - xy - 1 = 0$$

$$F_x = e^x - y, \quad F_y = \cos y - x$$

$$\left. \frac{dy}{dx} \right|_{x=0} = - \left. \frac{F_x}{F_y} \right|_{x=0, y=0} = - \frac{e^x - y}{\cos y - x} \Big|_{x=0, y=0} = -1$$

$$\left. \frac{d^2 y}{dx^2} \right|_{x=0} = - \left. \frac{d}{dx} \left(\frac{e^x - y}{\cos y - x} \right) \right|_{x=0, y=0, y'=-1}$$

$$= - \frac{(e^x - y')(\cos y - x) - (e^x - y)(-\sin y \cdot y' - 1)}{(\cos y - x)^2} \Big|_{\substack{x=0 \\ y=0 \\ y'=-1}}$$

$$= -3$$

导数的另一种求法— 利用隐函数求导

$$\sin y + e^x - xy - 1 = 0, \quad y = y(x)$$

两边对 x 求导

$$\cos y \cdot y' + e^x - y - xy' = 0$$

两边再对 x 求导

$$\sin y \cdot (y')^2 + \cos y \cdot y'' + e^x - y' - y' - xy'' = 0$$

令 $x = 0$, 注意此时 $y = 0, y' = -1$

$$\left. \frac{d^2 y}{dx^2} \right|_{x=0} = -3$$

$$\begin{aligned} & \left. y' \right|_{x=0} \\ &= - \frac{e^x - y}{\cos y - x} \Big|_{(0,0)} \\ &= -1 \end{aligned}$$

定理2. 若函数 $F(x, y, z)$ 满足:

- ① 在点 $P(x_0, y_0, z_0)$ 的某邻域内具有连续偏导数;
- ② $F(x_0, y_0, z_0) = 0$;
- ③ $F_z(x_0, y_0, z_0) \neq 0$,

则方程 $F(x, y, z) = 0$ 在点 (x_0, y_0) 某一邻域内可唯一确定一个连续函数 $z = f(x, y)$, 满足 $z_0 = f(x_0, y_0)$, 并有连续偏导数

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

定理证明从略, 仅就求导公式推导如下:

设 $z = f(x, y)$ 是方程 $F(x, y, z) = 0$ 所确定的隐函数,

$$F(x, y, f(x, y)) \equiv 0$$

将上式两端分别对 x 和 y 求导,

$$F'_x + F'_z \frac{\partial z}{\partial x} = 0, \quad F'_y + F'_z \frac{\partial z}{\partial y} = 0.$$

↓ 在 (x_0, y_0, z_0) 的某邻域内 $F'_z \neq 0$

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z}, \quad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}.$$

注 对于 $F(x_1, x_2, \dots, x_n, z) = 0$ 所确立的

$$z = z(x_1, x_2, \dots, x_n),$$

$$\frac{\partial z}{\partial x_i} = -\frac{F_{x_i}}{F_z} \quad i = 1, 2, \dots, n.$$

例 2 设 $x^2 + y^2 + z^2 - 4z = 0$, 求 $\frac{\partial^2 z}{\partial x^2}$.

解 令 $F(x, y, z) = x^2 + y^2 + z^2 - 4z$,

$$\text{则 } F_x = 2x, \quad F_z = 2z - 4, \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{x}{2-z},$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{(2-z) + x \frac{\partial z}{\partial x}}{(2-z)^2} = \frac{(2-z) + x \cdot \frac{x}{2-z}}{(2-z)^2} \\ &= \frac{(2-z)^2 + x^2}{(2-z)^3}. \end{aligned}$$

或者在 $x^2 + y^2 + z^2 - 4z = 0$ 两边对 x 求偏导

$$2x + 2z \frac{\partial z}{\partial x} = 4 \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{x}{2-z},$$

例3 设 $\phi(u, v)$ 具有连续的偏导数, 证明由方程 $\phi(cx - az, cy - bz) = 0$ 确定的函数 $z = z(x, y)$, 满足
$$a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = c.$$

证明 方法一 利用复合函数求导法则

方程的两端对 x 求导有

$$\phi_u \cdot (c - a \frac{\partial z}{\partial x}) + \phi_v \cdot (-b \frac{\partial z}{\partial x}) = 0$$

可得
$$\frac{\partial z}{\partial x} = \frac{c \phi_u}{a \phi_u + b \phi_v}.$$

$$\phi(cx - az, cy - bz) = 0$$

方程两端对 y 求偏导有

$$\phi_u \left(-a \frac{\partial z}{\partial y} \right) + \phi_v \left(c - b \frac{\partial z}{\partial y} \right) = 0$$

$$\text{可得} \quad \frac{\partial z}{\partial y} = \frac{c \phi_v}{a \phi_u + b \phi_v}.$$

于是有

$$a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = \frac{ac \phi_u + bc \phi_v}{a \phi_u + b \phi_v} = c.$$

方法二 公式法

记 $\phi(cx-az, cy-bz)=F(x, y, z)$, 则
 $F_x=c\phi_u, F_y=c\phi_v, F_z=-a\phi_u-b\phi_v$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{c\phi_u}{a\phi_u + b\phi_v},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{c\phi_v}{a\phi_u + b\phi_v},$$

所以
$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = \dots = c.$$

方法三 利用全微分形式的不变性

$$\begin{aligned}d\phi(cx-az, cy-bz) &= \phi_u d(cx-az) + \phi_v d(cy-bz) \\&= \phi_u (cdx-adz) + \phi_v (cdy-bdz) = 0\end{aligned}$$

移项 $c\phi_u dx + c\phi_v dy = (a\phi_u + b\phi_v)dz$

$$dz = \frac{c\phi_u}{a\phi_u + b\phi_v} dx + \frac{c\phi_v}{a\phi_u + b\phi_v} dy.$$

所以 $\frac{\partial z}{\partial x} = \frac{c\phi_u}{a\phi_u + b\phi_v}, \quad \frac{\partial z}{\partial y} = \frac{c\phi_v}{a\phi_u + b\phi_v}$

于是 $a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} = \dots = c.$

7.4.2、 方程组的情形

隐函数存在定理还可以推广到方程组的情形.

以两个方程确定两个隐函数的情况为例, 即

$$\begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases} \longrightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

由 F 、 G 的偏数组成的行列式

$$J = \frac{\partial (F, G)}{\partial (u, v)} = \begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}$$

称为 F 、 G 的雅可比 行列式.

$$\frac{\partial u}{\partial x} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(x, v)} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}},$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(u, x)} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}},$$

$$\frac{\partial u}{\partial y} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(y, v)} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}},$$

$$\frac{\partial v}{\partial y} = -\frac{1}{J} \frac{\partial(F, G)}{\partial(u, y)} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}.$$

仅就公式作如下推导:

设方程组 $\begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases}$ 有隐函数组 $\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$,

则 $\begin{cases} F(x, y, u(x, y), v(x, y)) \equiv 0 \\ G(x, y, u(x, y), v(x, y)) \equiv 0 \end{cases}$

两边对 x 求导得 $\begin{cases} F_x + F_u \cdot \frac{\partial u}{\partial x} + F_v \cdot \frac{\partial v}{\partial x} = 0 \\ G_x + G_u \cdot \frac{\partial u}{\partial x} + G_v \cdot \frac{\partial v}{\partial x} = 0 \end{cases}$

这是关于 $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ 的线性方程组, 在点 P 的某邻域内

系数行列式 $J = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} \neq 0,$

故得 $\frac{\partial u}{\partial x} = - \frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = - \frac{1}{J} \frac{\partial(F, G)}{\partial(x, v)}$

$$\frac{\partial v}{\partial x} = - \frac{1}{J} \frac{\partial(F, G)}{\partial(u, x)}$$

同样可得 $\frac{\partial u}{\partial y} = - \frac{1}{J} \frac{\partial(F, G)}{\partial(y, v)} \quad \frac{\partial v}{\partial y} = - \frac{1}{J} \frac{\partial(F, G)}{\partial(u, y)}$

例 4 设 $xu - yv = 0$, $yu + xv = 1$,

求 $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ 和 $\frac{\partial v}{\partial y}$.

解1 直接代入公式;

解2 运用公式推导的方法,

将所给方程的两边对 x 求导并移项

$$\begin{cases} x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x} = -u \\ y \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x} = -v \end{cases}, J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2,$$

注 特别 $\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$ 在一定条件下, 确定了

$y = y(x), z = z(x)$, 要求 $\frac{dy}{dx}, \frac{dz}{dx}$

条件: (1) F, G 连同它们的一切偏导在 (x_0, y_0, z_0) 的领域内连续。

$$(2) \quad F(x_0, y_0, z_0) = 0, G(x_0, y_0, z_0) = 0$$

$$(3) \quad J \bigg|_{(x_0, y_0)} = \frac{\partial(F, G)}{\partial(y, z)} \bigg|_{(x_0, y_0)} \neq 0$$

证略。求法

对 $\begin{cases} F(x, y(x), z(x)) = 0 \\ G(x, y(x), z(x)) = 0 \end{cases}$ 两边对 x 求导:

$$\begin{cases} F_x + F_y \frac{dy}{dx} + F_z \frac{dz}{dx} = 0 \\ G_x + G_y \frac{dy}{dx} + G_z \frac{dz}{dx} = 0 \end{cases}$$

从中解出 $\frac{dy}{dx}, \frac{dz}{dx}$

例5
$$\begin{cases} z = x^2 + y^2 \\ x^2 + 2y^2 + 3z^2 = 20 \end{cases}, \text{求 } \frac{dy}{dx}, \frac{dz}{dx}$$

解 运用公式推导的方法,

将所给方程的两边对 x 求导

$$\begin{cases} \frac{dz}{dx} = 2x + 2y \frac{dy}{dx} \\ 2x + 4y \frac{dy}{dx} + 6z \frac{dz}{dx} = 0 \end{cases},$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x(6z+1)}{2y(3z+1)}, \frac{dz}{dx} = \frac{z-x}{x-y}.$$

例6 $x = u + v, y = u - v, z = u^2 v^2$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

解 方程组确定三个二元函数

$$\begin{cases} u = u(x, y) \\ v = v(x, y), \\ z = z(x, y) \end{cases}$$

在三个方程两边分别对x求偏导

$$\begin{cases} 1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}, \\ 0 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}, \\ \frac{\partial z}{\partial x} = 2uv^2 \frac{\partial u}{\partial x} + 2vu^2 \frac{\partial v}{\partial x}, \end{cases} \quad \text{解得:} \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{1}{2}, \\ \frac{\partial z}{\partial x} = uv^2 + vu^2 = \frac{x(y^2 - x^2)}{4} \end{cases}$$

在三个方程两边分别对 y 求偏导

$$\begin{cases} 0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}, \\ 1 = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}, \\ \frac{\partial z}{\partial y} = 2uv^2 \frac{\partial u}{\partial y} + 2vu^2 \frac{\partial v}{\partial y}, \end{cases}$$

$$\text{解得} \begin{cases} \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} = \frac{1}{2}, \\ \frac{\partial z}{\partial y} = uv^2 - vu^2 = \frac{y(y^2 - x^2)}{4} \end{cases}$$

